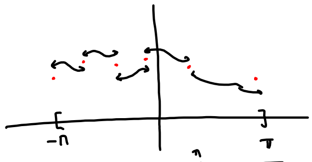
Recall:

D = space of all piecewise continuous function with one-sided limits



Inner product: $\langle f,g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$

Norm: $\|f\|_2 = \langle f,f\rangle^{1/2} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx\right)^{1/2}$

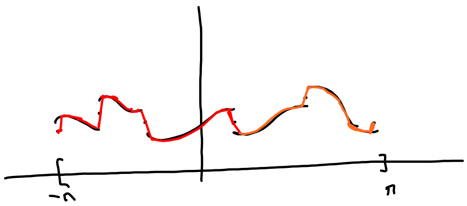
$C_p = \{f : [-\pi,\pi] \to \mathbb{C} : f$ is continuous and $f(\pi) = f(-\pi)\}$

Trigonometric polynomial: $\sum_{n=-N}^{N} \alpha_n e_n$

Prop: The trigonometric polynomials are dense in C_p in the L^2-norm; i.e. for any $f \in C_p$ and any $\varepsilon > 0$, there is a trigonometric polynomial p such $\|f - p\|_2 < \varepsilon$

Proposition: C_p is dense in D in $\|\cdot\|_2$-norm.

Sketch of Proof: Given an $f \in D$ and an $\varepsilon > 0$, we must show that there is a $g \in C_p$ such that $\|f - g\|_2 < \varepsilon$



Theorem: Let $f \in D$ and define

$$\alpha_n = \langle f, e_n\rangle \text{ for } n \in \mathbb{Z}.$$

Then

$$f = \sum_{n=-\infty}^{\infty} \alpha_n e_n \text{ in the } \|\cdot\|_2\text{-norm; i.e.}$$

$$\|f - \sum_{n=-N}^{N} \alpha_n e_n\|_2 \to 0 \text{ as } N \to \infty.$$

Proof: Given $\varepsilon > 0$, we must show that there is a $N \in \mathbb{N}$ such that if $M \geq N$, then

$$\|f - \sum_{n=-M}^{M} \alpha_n e_n\|_2 < \varepsilon.$$

By the previous result, there is $g \in C_p$ such that

$$\|f - g\|_2 < \frac{\varepsilon}{2}.$$

By what we proved last time, there is a trigonometric polynomial p such

$$\|g - p\|_2 < \frac{\varepsilon}{2}.$$

By the triangle inequality

$$\|f - p\|_2 = \|(f-g) + (g-p)\|_2 \leq \|f-g\|_2 + \|g-p\|_2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

If N is the degree of p; i.e. $\sum_{n=-N}^{N} \beta_n e_n$, then for any $M \geq N$,

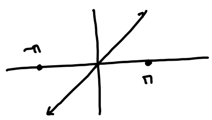
$$p \in \operatorname{Span}\{e_{-M}, \dots, e_M\}$$

But since $\sum_{n=-M}^{M} \alpha_n e_n$ is the element in $\operatorname{Sp}\{e_{-M}, \dots, e_M\}$ that is closest to f in $\|\cdot\|_2$-norm, we have

$$\|f - \sum_{n=-M}^{M} \alpha_n e_n\|_2 \leq \|f - p\|_2 < \varepsilon.$$

But what does this mean: Last time we computed the Fourier series of x

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx.$$



Hence

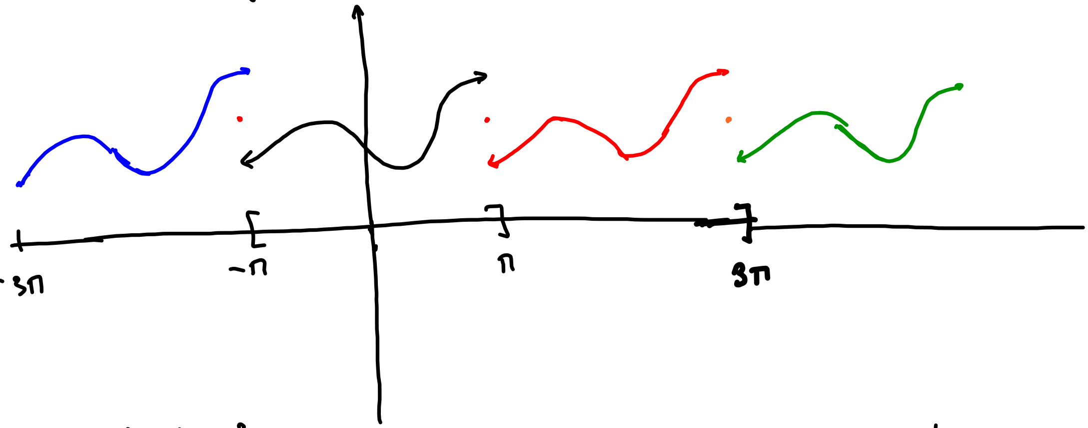
$$\|x - \sum_{n=1}^{N} \frac{2(-1)^{n+1}}{n} \sin nx\|_2 \to 0$$

but do we have

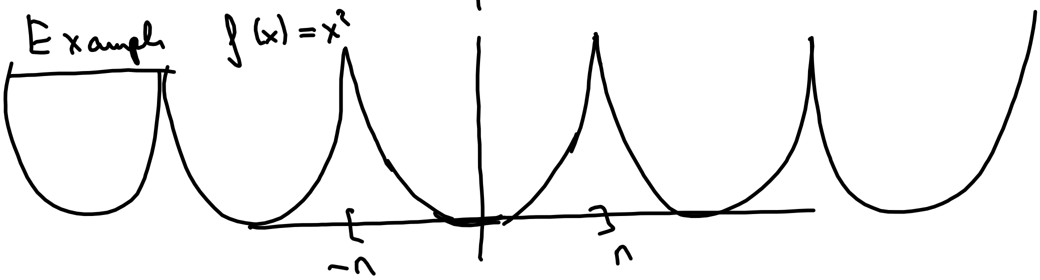
$$x = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{2(-1)^n}{n} \sin x$$

Dirichlet kernel (10.3)

Periodic extension $f(x+2k\pi) = f(x)$

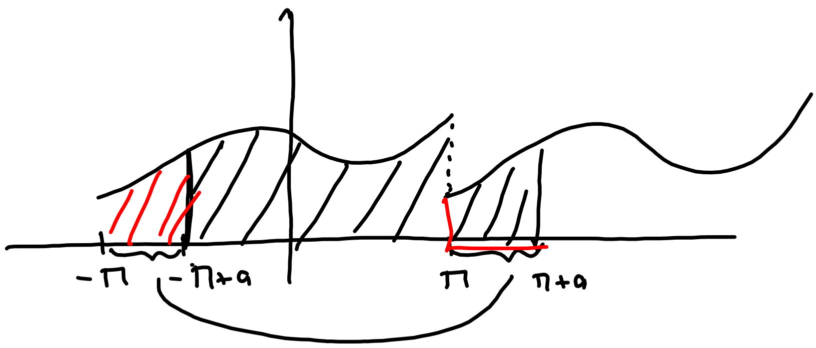


Example $f(x) = x^2$



Why: $\int_{-\pi}^{\pi} f(x)\,dx$ $\qquad u = x+a \Rightarrow x = u-a$, $du = du$

$$= \int_{-\pi+a}^{\pi+a} f(u-a)\,du = \int_{-\pi}^{\pi} f(u-a)\,du .$$



Geometric series: $a_0 + a_0 r + a_0 r^2 + \dots + a_0 r^n = a_0 \frac{1-r^{n+1}}{1-r}$

$$= \frac{a_0 - a_0 r^{n+1}}{1-r} = \frac{\text{first term in series} - \text{first term not in series}}{1-r}$$

Fourier approximations to f: $\sum_{n=-N}^{N} a_n e_n$

$$\sum_{n=-N}^{N} a_n e_n(x) = \sum_{n=-N}^{N} \langle f, e_n\rangle e_n(x) = \sum_{n=-N}^{N}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt\right) e^{inx}$$

$$= \frac{1}{2\pi}\sum_{n=-N}^{N}\int_{-\pi}^{\pi} f(t)\, \underbrace{e^{inx} e^{-int}}\, dt = \frac{1}{2\pi}\sum_{n=-N}^{N}\int_{-\pi}^{\pi} f(t)\, \underbrace{e^{in(x-t)}}_{\text{periodic}}\, dt$$

$u = x-t$

$t = x-u$

$dt = du$

$$= \frac{1}{2\pi}\sum_{n=-N}^{N}\int_{-\pi}^{\pi} f(x-u)\, e^{inu}\, du$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\, \underbrace{\sum_{n=-N}^{N} e^{inu}}\, du \longleftarrow$$

$D_N(u)$ = the Dirichlet kernel.

Let us take a look at $D_N(u)$:

$$D_N(u) = \sum_{n=-N}^{N} (e^{iu})^n \leftarrow \text{ geometric series with quotient } e^{iu}$$

$$= \frac{e^{-iNu} - e^{i(N+1)u}}{1-e^{iu}} = \frac{\cancel{e^{i\frac{u}{2}}}\left(e^{-i(N+\frac12)u} - e^{i(N+\frac12)u}\right)}{\cancel{e^{i\frac{u}{2}}}\left(e^{-i\frac{u}{2}} - e^{i\frac{u}{2}}\right)}$$

$$\boxed{\sin x = \frac{e^{ix} - e^{-ix}}{2i}}$$

$$= \frac{\dfrac{e^{i(N+\frac12)u} - e^{-i(N+\frac12)u}}{2i}}{\dfrac{e^{i\frac{u}{2}} - e^{-i\frac{u}{2}}}{2i}}$$

$$= \frac{\sin\left[(N+\frac12)u\right]}{\sin\frac{u}{2}} \quad \text{for } u \neq 0$$

This means

$$D_N(u) = \begin{cases} \dfrac{\sin\left[(N+\frac12)u\right]}{\sin\frac{u}{2}} & \text{for } u \neq 0 \\ 2N+1 & \text{for } u = 0. \end{cases}$$

Proposition: The Dirichlet kernel is given by

$$D_N(u) = \sum_{n=-N}^{N} e^{inu} = \begin{cases} \dfrac{\sin[(N+\frac{1}{2})u]}{\sin\frac{u}{2}} & \text{for } u \neq 0 \\ 2N+1 & \text{for } u = 0. \end{cases}$$

It has the following properties.

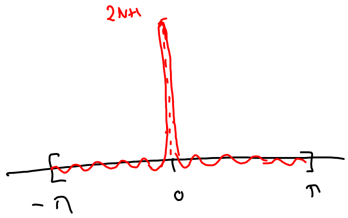
(i) $D_N(u) \leq 2N+1$ with equality for $u=0$.

(ii) D_N is a even function, $D_N(-u) = D_N(u)$

(iii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(u)\,du = 1$

(iv) $\int_{-\pi}^{\pi} |D_N(u)|\,du \to \infty$.

(v) The graph looks something like this

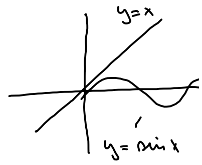


Proof: (iii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(u)\,du = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=-N}^{N} e^{inu}\,du$

$$= \sum_{n=-N}^{N} \underbrace{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inu}\,du\right)}_{\substack{= \\ 0 \text{ if } n\neq 0 \\ 1 \text{ if } n=0}} = \underline{\underline{1}}$$

(iv) Want to prove that

$$\int_{-\pi}^{\pi} \left|\frac{\sin[(N+\frac{1}{2})u]}{\sin\frac{u}{2}}\right| du \to \infty \quad \text{as } N\to\infty$$

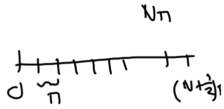


$$= 2\int_0^{\pi} \frac{|\sin[(N+\frac{1}{2})u]|}{\sin\frac{u}{2}}\,du$$

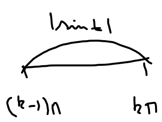
$$\geq 2\int_0^{\pi} \frac{|\sin[(N+\frac{1}{2})u]|}{\frac{u}{2}}\,du \qquad \frac{\sin x \leq x}{\text{for } x \geq X}$$

$$= 4\int_0^{\pi} \frac{|\sin[(N+\frac{1}{2})u]|}{u}\,du \qquad \frac{t = (N+\frac{1}{2})u}{dt = (N+\frac{1}{2})\,du}$$

$$= 4\int_0^{(N+\frac{1}{2})\pi} \frac{|\sin t|}{\frac{t}{\cancel{N+\frac{1}{2}}}}\,\frac{dt}{\cancel{N+\frac{1}{2}}}$$



$$\geq 4\sum_{k=1}^{N}\int_{(k-1)\pi}^{k\pi} \frac{|\sin t|}{k\pi}\,dt$$

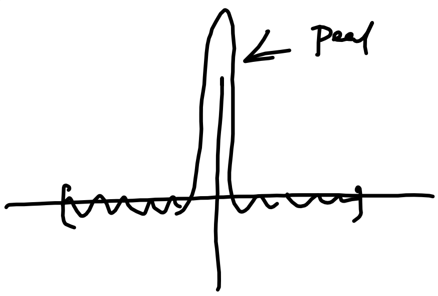


$$= \frac{4}{\pi}\sum_{k=1}^{N}\frac{1}{k}\underbrace{\int_{(k-1)\pi}^{k\pi} |\sin t|\,dt}_{2}$$

$$= \frac{8}{} \sum^{N} \frac{1}{} \to \infty \quad \text{as } N\to\infty$$

Recall that

$$f(x) \longleftarrow \sum_{n=-N}^{N} \alpha_n e_n(x) = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u) D_N(u)\,du}_{\downarrow\, f(x)}$$



$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(u)\,du = 1$$