Completeness

Def. A sequence $\{x_n\}$ of points in a metric space is a Cauchy sequence if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that whenever $n, k \geq N$, then $d(x_n, x_k) < \varepsilon$.

Prop: All convergent sequences are Cauchy sequences.

Given an $\varepsilon > 0$, I have to find an N such $d(x_n, x_m) < \varepsilon$ when $n, m \geq N$. Since $x_n \to x$, there is $N \in \mathbb{N}$ such that when $n \geq N$, $d(x_n, x) < \frac{\varepsilon}{2}$. If $n, m \geq N$, then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Def: A metric space X is complete if all Cauchy sequences converge (to a point $x \in X$)

Examples: $\mathbb{R}^d$ is complete

$\mathbb{Q}$ is not complete

Assume $A \subseteq X$, let d_A be the metric on A defined by restriction: $d_A(a, b) = d(a, b)$.

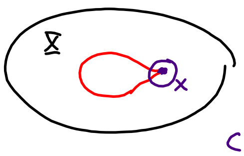
Proposition: Assume that (X, d) is a complete metric space and that $A \subseteq X$ (nonempty). Then (A, d_A) is complete if and only if A is closed.

Example: $X = \mathbb{R}$ $[0,1]$ is complete

$(0,1)$ is not complete.

Proof: Assume that A is closed and that $\{x_n\}$ is a Cauchy sequence in A. Then $\{x_n\}$ is a Cauchy sequence in X, and since X is complete it will converge to a point $x \in X$. Since A is closed, x is in A.

Assume that A is not closed. Then there is a boundary point x that does not belong to A.



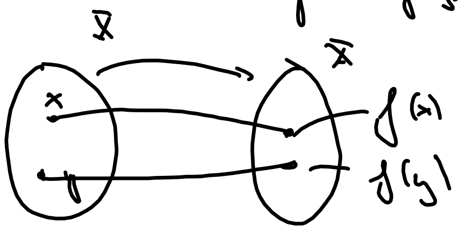
For all n, there is an element $x_n \in A$ in the ball $B(x, \frac{1}{n})$. But then $x_n \to x$ in X, and hence $\{x_n\}$ is a Cauchy sequence in X. This means that $\{x_n\}$ is a Cauchy sequence in A, but it does not converge to a point in A.

$d(x_n, x_m) < \varepsilon \qquad \rightarrow X$

Def: Assume that $f: X \to X$. A <u>fixed point</u> for f is just a point $a \in X$ such that $\underbrace{f(a) = a}_{f(x) = x}$

Def: Assume that (X, d) is a metric space. A function $f: X \to X$ is called a <u>contraction</u> if there is a number $0 \le s < 1$ such that

$$d(f(x), f(y)) \le s\, d(x,y)$$



Assume $x_0 \in X$

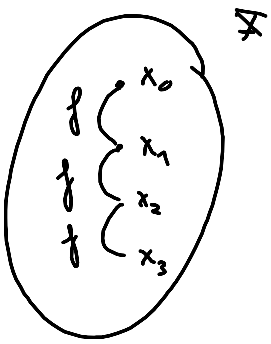
$x_1 = f(x_0)$

$x_2 = f(x_1) = f(f(x_0)) = f^{\circ 2}(x_0)$

$x_3 = f(x_2) = f(f^{\circ 2}(x_0)) = f^{\circ 3}(x_0)$

$\vdots$

$x_n = f(x_{n-1}) = f^{\circ n}(x_0)$



<u>Observation:</u> $d(f^{\circ n}(x), f^{\circ n}(y))$

$d(f(x), f(y)) \le \underline{s\, d(x,y)}$

$d(f^{\circ 2}(x), f^{\circ 2}(y)) = d(f(f(x)), f(f(y))) \le s\, d(f(x), f(y))$

$\le s\, s\, d(x,y) = s^2 d(x,y)$

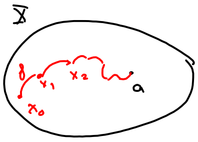
$\vdots$

$d(f^{\circ n}(x), f^{\circ n}(y)) \le s^n d(x,y)$

Banach's Fixed Point Theorem: Assume (X,d) is a complete metric space and that $f: X\to X$ is a contraction. Then f has a unique fixed point a. Moreover, if $x_0 \in X$, the sequence $\{x_n\}$ where

$$x_0,\ x_1 = f(x_0),\ x_2 = f^{\circ 2}(x_0), \dots, x_n = f^{\circ n}(x_0), \dots$$

converges to a.



Proof: Let us first show that there can only be one fixed point. Assume that a, b are fixed points. Then

$$d(a,b) = d(\underset{a}{f(a)}, \underset{b}{f(b)}) \le s\, d(a,b)$$

which is only possible if $d(a,b)=0$, i.e. $a=b$.

Pick a point $x_0 \in X$ and consider

$$x_0,\ x_1 = f(x_0),\ x_2 = f^{\circ 2}(x_0), \dots, x_n = f^{\circ n}(x_0), \dots$$

Let us show that $\{x_n\}$ is a Cauchy sequence.

$$d(x_n, x_{n+k}) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{n+k-1}, x_{n+k})$$

$$= d(f^{\circ n}(x_0), f^{\circ n}(x_1)) + d(f^{\circ (n+1)}(x_0), f^{\circ (n+1)}(x_1)) + \dots + d(f^{\circ (n+k-1)}(x_0), f^{\circ (n+k-1)}(x_1))$$

$$\le s^n d(x_0,x_1) + s^{n+1} d(x_0,x_1) + \dots + s^{n+k-1} d(x_0,x_1)$$

$$\le d(x_0,x_1)\Big(\underbrace{s^n + s^{n+1} + \dots + s^{n+k-1} + s^{n+k} + \dots}_{\text{infinite geometric series}}\Big)$$

$$= d(x_0,x_1)\frac{s^n}{1-s} \longrightarrow 0 \text{ as } n\to\infty.$$

Given $\varepsilon > 0$, there is an $N\in\mathbb{N}$ such that when $n \ge N$, then $d(x_0,x_1)\frac{s^n}{1-s} < \varepsilon$. But then $n, n+k \ge N$, then

$$d(x_n, x_{n+k}) \le d(x_0,x_1)\frac{s^n}{1-s} < \varepsilon.$$

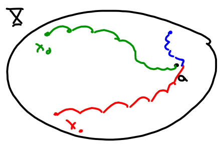
This means that $\{x_n\}$ is a Cauchy sequence and hence converges since X is complete. Let $a = \lim_{n\to\infty} x_n$. Note that a is a fixed point because

$$x_{n+1} \to a$$

$$\overset{\|}{f}(\underset{\downarrow a}{x_n}) \to f(a)$$

Hence $a = f(a)$, which means that a is a fixed point.

As there is only one fixed point, all such sequences converge to the same point.



Corollary: For any start point x_0,

$$d(\underset{f^{\circ n}(x_0)}{\underset{\|}{x_n}}, a) \le d(x_0,x_1)\frac{s^n}{1-s}$$

Proof: Recall that

$$d(x_n, \overset{a}{\overset{\uparrow}{x_{n+k}}}) \le d(x_0,x_1)\frac{s^n}{1-s}$$

Keep n fixed, and let $k\to\infty$

$$d(x_n,a) \le d(x_0,x_1)\frac{s^n}{1-s}.$$

Compactness

Starting point:

Bolzano-Weierstrass: Any bounded sequence in $\mathbb{R}^d$ has a convergent subsequence.

Corollary: If A is a closed and bounded subset of $\mathbb{R}^d$, then every sequence from A has a subsequence that converges to a point in A.

Proof: Let $\{x_n\}$ be a sequence in A. Since A is bounded, $\{x_n\}$ is bounded and hence has a convergent subsequence according BW. Since A is closed, the limit has to be in A, which is what we had to prove.

Definition: Assume that (X, d) is a metric space. A subset $K \subseteq X$ is called compact if every sequence from K has a subsequence converging to a point in K.

Observation: Closed and bounded subsets of $\mathbb{R}^d$ are compact.

Proposition: All compact sets are closed.

Proof: We prove that if a set K is not closed, then it is not compact. Since K is not closed, there is a boundary point x that does not belong to K, and hence a sequence $\{x_n\}$ of points from K converging to x. All subsequences of $\{x_n\}$ converge to x as well, and hence they can not converge to a point in K.