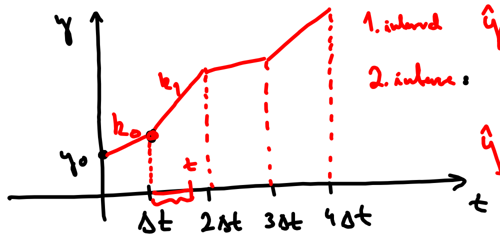
Back to differential equations

$$y'(t) = f(t, y(t)), \quad y(0) = y_0$$

Equiv:

$$y(t) = y_0 + \int_0^t f(s, y(s))\, ds$$

Euler's method:



$$y'(0) = f(0, y_0) = k_0$$

1. interval: $\hat{y}(t) = y_0 + k_0 t = y_0 + \int_0^t f(0, y_0)\, ds$

2. interval: $\hat{y}'(\Delta t) = f(\Delta t, \hat{y}(\Delta t)) = k_1$

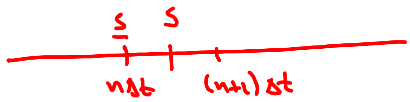
$$\hat{y}(t) = y_0 + \int_0^{\Delta t} f(0, y_0)\, ds + k_1 (t - \Delta t)$$

$$= y_0 + \int_0^{\Delta t} f(0, y_0)\, ds + \int_{\Delta t}^{t} f(\Delta t, \hat{y}(\Delta t))\, ds$$

n-th interval: $n\Delta t \le t \le (n+1)\Delta t$

$$\hat{y}(t) = y_0 + \int_0^{\Delta t} f(0, y_0)\, ds + \int_{\Delta t}^{2\Delta t} f(\Delta t, \hat{y}(\Delta t))\, ds + \int_{2\Delta t}^{3\Delta t} f(2\Delta t, \hat{y}(2\Delta t))\, ds + \dots + \int_{n\Delta t}^{t} f(n\Delta t, \hat{y}(n\Delta t))\, ds$$

Notation: $\underline{s}$



$$\hat{y}(t) = y_0 + \int_0^t f(\underline{s}, \hat{y}(\underline{s}))\, ds$$

$$\underbrace{\hat{y}(t) = y_0 + \int_0^t f(s, \hat{y}(s))\, ds}_{\hat{y} \text{ solution!}} + \underbrace{\int_0^t [f(\underline{s}, \hat{y}(\underline{s}))\, ds - f(s, \hat{y}(s))]\, ds}_{\text{How much } \hat{y} \text{ deviates from being a solution.}}$$

Strategy $\Delta t \to 0$

$$y(t) = y_0 + \int_0^t f(s, y(s))\, ds \qquad 0$$

<u>Proposition:</u> Let $a>0$ and assume that $f:[0,a]\times\mathbb{R}\to\mathbb{R}$ is continuous. If $\{y_n\}$ is a sequence of continuous functions $y_n:[0,a]\to\mathbb{R}$ converging uniformly to y on $[0,a]$, then the functions

$$I_n(t)=\int_0^t f(s,\underline{y_n(s)})\,ds \qquad t\in[0,a]$$

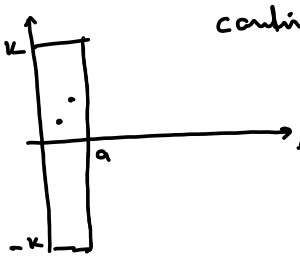
converge uniformly on $[0,a]$ to

$$I(t)=\int_0^t f(s,y(s))\,ds$$

<u>Proof:</u> Since $y_n\to y$ uniformly, the sequence $\{y_n\}$ is bounded in the sense that there is $K\in\mathbb{R}$ such that

$$\underline{|y_n(t)|}\le K \text{ for all } n \text{ and all } t.$$

Since the set $[0,a]\times[-K,K]\subseteq\mathbb{R}^2$ is compact, f is uniformly continuous on $[0,a]\times[-K,K]$. Hence given an $\varepsilon>0$, there is a $\underline{\delta}$ such that if



$$|(s,y)-(s',y')|<\underline{\delta}, \text{ then}$$

$$|f(s,y)-f(s',y')|<\frac{\varepsilon}{a}$$

Since $y_n\to y$ uniformly, there is $N\in\mathbb{N}$ such that if $\underline{n\ge N}$, then $\underline{|y_n(t)-y(t)|<\delta}$ for all t.

But this means that for $\underline{n\ge N}$, we have

$$|I_n(t)-I(t)|=\left|\int_0^t f(s,y_n(s))\,ds-\int_0^t f(s,y(s))\,ds\right|$$

$$\le\int_0^t |f(\underbrace{s,y_n(s))-f(s,y(s)}_{\le\delta})|\,ds \le \int_0^t \frac{\varepsilon}{a}\,ds=\frac{\varepsilon}{a}t\le\frac{\varepsilon}{a}\cdot a=\underline{\varepsilon}$$

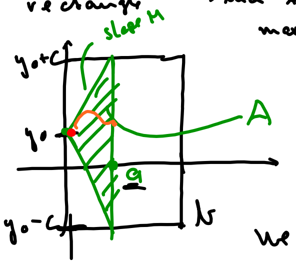
Thus I_n converges uniformly to I.

Theorem: Assume that $f: [0,\infty)\times\mathbb{R}\to\mathbb{R}$ is continuous and that $y_0\in\mathbb{R}$. Then there is an $a>0$ and a function $y:[0,a]\to\mathbb{R}$ such that

$$y'(t) = f(t, y(t)) \text{ for all } t\in[0,a]$$

and $y(0) = y_0$.

Proof: Choose (big) numbers c and b. Look at the rectangle. Since the rectangle is compact, $|f|$ has a maximal value M on this set.



$$|f(t,y)| \le M$$

Let $\hat{y}_n$ be the approximate solution we get when we apply Euler's method with stepsize $\frac{a}{n}$.

$$\hat{y}_n(t) = y_0 + \int_0^t f(s,\hat{y}_n(s))\,ds + \int_0^t [f(\underline{s},\hat{y}_n(\underline{s})) - f(s,\hat{y}_n(s))]\,ds$$

The sequence $\{\hat{y}_n\}$ is bounded (it lives within A) and equicontinuous since $|\hat{y}_n(t) - \hat{y}_n(s)| \le M|t-s|$.

By the Arzela-Ascoli Theorem the sequence $\{\hat{y}_n\}$ has a subsequence $\{\hat{y}_{n_k}\}$ converging uniformly to some function y.

Hence

$$\hat{y}_{n_k}(t) = y_0 + \int_0^t f(s,\hat{y}_{n_k}(s))\,ds + \underbrace{\int_0^t [f(\underline{s},\hat{y}_{n_k}(\underline{s})) - f(s,\hat{y}_{n_k}(s))]\,ds}_{a\varepsilon}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \text{Prop} \qquad \downarrow ??$$

$$y(t) = y_0 + \int_0^t f(s,y(s))\,ds \qquad\qquad 0$$

$\overbrace{\underline{s}\quad s}^{\frac{a}{n_k}}$

Hence the theorem will follow if we can prove ??

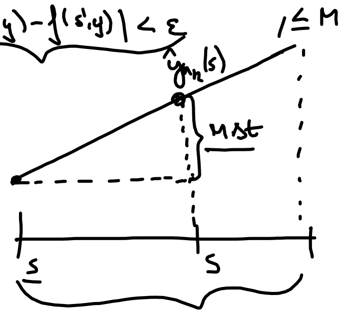
Since f is continuous, it is uniformly continuous on A, and hence for every $\varepsilon>0$, there is a $\delta>0$ such that if $|(s,y)-(s',y')|<\delta$, then $|f(s,y)-f(s',y')|<\varepsilon$.

But $|(\underline{s},\hat{y}_{n_k}(\underline{s})) - (s,\hat{y}_{n_k}(s))|$

$$\le \sqrt{(\underline{s}-s)^2 + (M\Delta t)^2}$$

$$= \sqrt{\Delta t^2 + M^2\Delta t^2} \le \Delta t\sqrt{1+M^2} < \delta$$

$\Delta t = \frac{a}{n_k}$; can get this less than δ by choosing n large



What about uniqueness and what about lifetime of solutions

Example: $y'(t) = 1+y(t)^2 \quad y(0) = 0$

$F(t,y) = 1+y^2$

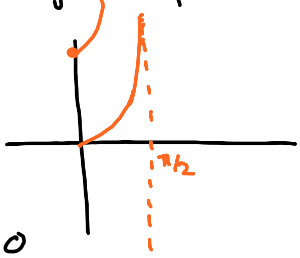
Separable: $\dfrac{y'}{1+y^2} = 1$

Integrate $\displaystyle\int \frac{y'}{1+y^2}\,dt = \int 1\,dt$ dy

$\arctan y = t + C \quad | \; y(0)=0$ forces $C=0$

$\arctan y = t$

$y = \tan t.$



Uniqueness: $y' = \frac{2}{3} y^{1/3} \quad y(0)=0$

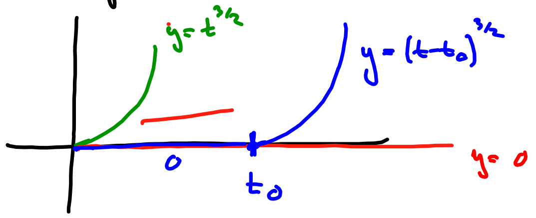
Obvious solution: $y(t)=0.$

$f(t,y) = \frac{2}{3} y^{1/3}$

$y^{-1/3} y' = \frac{2}{3}$

$\frac{2}{3} y^{2/3} = \frac{2}{3} t + \cancel{C}$ (initial condition forces $C=0$)

$y^{2/3} = t \Rightarrow y(t) = t^{3/2}$



infinitely