Integration

$\vec{F} : [a,b] \to V$ where V is a complete, normed space. Want to define $\int_a^b \vec{F}(x)\,dx$. Idea: Riemann sums

$$\int_a^b \vec{F}(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\vec{F}(c_i)(x_i - x_{i-1})}$$

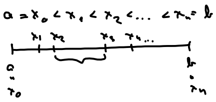
makes sense since V is a vector space.

defined since V is a normed space.

has a decent chance to exist since V is complete.

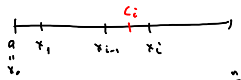
Partition of $[a,b]$: $\Pi = \{x_0, x_1, \dots, x_n\}$ such that

$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$



Mesh of Π: $|\Pi| = \max\{x_i - x_{i-1} : i = 1, \dots, n\}$

Selection S of Π: $S = \{c_1, c_2, \dots, c_n\}$ where $c_i \in [x_{i-1}, x_i]$

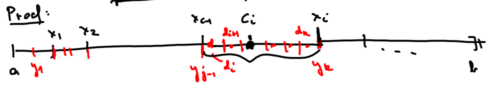


Riemann sum of $\vec{F}, \Pi, S$: $R(\vec{F}, \Pi, S) = \sum_{i=1}^{n} \vec{F}(c_i)(x_i - x_{i-1})$

Proposition: Assume that Π is a partition of $[a,b]$ and that $\hat{\Pi}$ is a finer partition of $[a,b]$ (i.e. $\Pi \subseteq \hat{\Pi}$). Assume also that if c, d are in the same partition interval $c, d \in [x_{i-1}, x_i]$ for Π, then $\|\vec{F}(c) - \vec{F}(d)\| \le M$. Then for any selection S of Π and $\hat{S}$ of $\hat{\Pi}$, we have

$$\|R(\vec{F}, \Pi, S) - R(\vec{F}, \hat{\Pi}, \hat{S})\| \le M\|b-a\|$$

Proof:



$$\|F(c_i)(x_i - x_{i-1}) - \sum_{l=j}^{k} F(d_l)(y_l - y_{l-1})\|$$

$$= \| \sum_{l=j}^{k} F(c_i)(y_l - y_{l-1}) - \sum_{l=j}^{k} F(d_l)(y_l - y_{l-1}) \|$$

$$\le \sum_{l=j}^{k} \|F(c_i) - F(d_l)\|(y_l - y_{l-1})$$

$$\le \sum_{l=j}^{k} M(y_l - y_{l-1}) = M(x_i - x_{i-1})$$

Hence

$$\|R(\vec{F}, \Pi, S) - R(\vec{F}, \hat{\Pi}, \hat{S})\| \le M \sum (x_i - x_{i-1}) = M(b-a)$$

Proposition: Assume that $\vec{F} : [a,b] \to V$ is continuous. Then for a given $\varepsilon > 0$, there is a δ such that if Π and Π' are two partitions with mesh less than δ, then

$$\|R(\vec{F}, \Pi, S) - R(\vec{F}, \Pi', S')\| < \varepsilon$$

for all selections S and S'.

Proof: Since $[a,b]$ is compact, $\vec{F}$ is uniformly continuous, and hence given an $\varepsilon > 0$, I can find a $\delta > 0$ such that if $|u-v| < \delta$, then

$$\|\vec{F}(u) - \vec{F}(v)\| < \frac{\varepsilon}{2(b-a)}$$

Let Π and Π' be two partitions with mesh less than δ, and let $\hat{\Pi} = \Pi \cup \Pi'$ be the common refinement. But then by the previous result

$$\|R(\vec{F}, \Pi, S) - R(\vec{F}, \hat{\Pi}, \hat{S})\| < \underbrace{\frac{\varepsilon}{2(b-a)}}_{M}(b-a) = \frac{\varepsilon}{2}$$

$$\|R(\vec{F}, \Pi', S') - R(\vec{F}, \hat{\Pi}, \hat{S})\| < \frac{\varepsilon}{2}.$$

By the triangle inequality

$$\|R(\vec{F}, \Pi, S) - R(\vec{F}, \Pi', S')\| = \|R(\vec{F}, \Pi, S) - R(\vec{F}, \hat{\Pi}, \hat{S}) + R(\vec{F}, \hat{\Pi}, \hat{S}) - R(\vec{F}, \Pi', S')\|$$

$$\le \|R(\vec{F}, \Pi, S) - R(\vec{F}, \hat{\Pi}, \hat{S})\| + \|R(\vec{F}, \hat{\Pi}, \hat{S}) - R(\vec{F}, \Pi', S')\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Observe: Assume that we have sequence (Π_n, S_n) such that $|\Pi_n| \to 0$. Then the $\{R(\vec{F}, \Pi_n, S_n)\}$ is a Cauchy sequence, and will converge since V is complete. Also note that if $(\hat{\Pi}_m, \hat{S}_m)$ then $\{R(F, \hat{\Pi}_m, \hat{S}_m)\}$ will converge to the same limit.

Definition: We define

$$\int_a^b \vec{F}(x)\,dx = \lim_{n\to\infty} R(\vec{F}, \Pi_n, S_n)$$

where (Π_n, S_n) is any sequence such $|\Pi_n| \to 0$.

Properties

Proposition: The integral is linear in the sense that

$$\int_a^b (\alpha \vec{F} + \beta \vec{G})\,dx = \alpha \int_a^b \vec{F}\,dx + \beta \int_a^b \vec{G}\,dx \quad \text{for all } \alpha, \beta \in k.$$

Proposition. If $a < c < b$, then

$$\int_a^b \vec{F}(x)\,dx = \int_a^c \vec{F}(x)\,dx + \int_c^b \vec{F}(x)\,dx$$

Fundamental Theorem of Calculus, Part I: Assume that $\vec{F} : [a,b] \to V$ is continuous. Then the function

$$\vec{I}(x) = \int_a^x \vec{F}(t)\,dt$$

is differentiable and $\vec{I}'(x) = \vec{F}(x)$.

Proof: We have

$$\sigma(r) = \vec{I}(x+r) - \vec{I}(x) - \vec{F}(x)\cdot r$$

and need to prove that $\frac{\|\sigma(r)\|}{r} \to 0$ as $r \to 0$

$$\frac{\|\sigma(r)\|}{r} = \frac{\| \int_a^{x+r} \vec{F}(t)\,dt - \int_a^x \vec{F}(t)\,dt - \vec{F}(x)\cdot r \|}{r} \qquad \int_a^x + \int_x^{x+r}$$

$$= \frac{\| \int_x^{x+r} \vec{F}(t)\,dt - \vec{F}(x)\cdot r \|}{r}$$

$$= \frac{\| \int_x^{x+r} (\underbrace{\vec{F}(t) - \vec{F}(x)}_{\wedge|\ \varepsilon})\,dt \|}{r} \leq \frac{\varepsilon r}{r} \leq \varepsilon.$$

For any $\varepsilon > 0$, there is a δ such that if $|r| < \delta$, then $\|\vec{F}(t) - \vec{F}(x)\| < \varepsilon$ for all t between x and $x+r$.

<u>Fundamental Theorem of Calculus, Part II</u>: Assume that $\vec{F}: [a,b] \to V$ is continuous and that $\vec{F}'(t)$ exists for all $t \in (a,b)$. Then

$$\vec{F}(d) - \vec{F}(c) = \int_c^d \vec{F}'(t)\,dt \quad \text{for all } c,d \in [a,b).$$

<u>Proof</u>: Define $G(x) = \int_a^x \vec{F}'(t)\,dt$. Then by Part I,

$\underline{G'(x) = \vec{F}'(x)}$. Let $\vec{H}(x) = \vec{G}(x) - \vec{F}(x)$. Then

$\underline{\vec{H}'(x)} = \vec{G}'(x) - \vec{F}'(x) = \vec{F}'(x) - \vec{F}'(x) = 0$. Hence

$\|\vec{H}(d) - H(\vec{c})\| = 0$

Hence $G(x) = F(x)$ which is what we wanted to prove.

$\|H'(t)\| = 0 = g'(t)$.

$\|H(d) - H(c)\| \leq g(d) - g(c) = 0$

Mean Value Theorem

If $\|\vec{F}'(t)\| \leq g'(t)$, then

$\|\vec{H}(b) - \vec{H}(a)\| \leq g(b) - g(a)$

In this case $\|\vec{H}'(t)\| = 0$

Hence $g(t) = 0$ satisfies the condition, and thus

$\|\vec{H}(b) - H(a)\| \leq g(b) - g(a) = 0$

Taylor's formula

One-dimensional case:

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \dots + \frac{f^{(n)}(a)}{n!}h^n + \underbrace{R_n(h)}_{\text{remainder term}}$$

Normed space

$$\vec{F}(\vec{a}+\vec{h}) = \vec{F}(\vec{a}) + \underbrace{\dots\dots}_{\text{derivatives}} + \text{remainder term.}$$

Lemma: Assume that $\vec{F}:[0,1]\to V$ is $n+1$-times differentiable. Let

$$G(t) = \sum_{k=0}^{n} \frac{(1-t)^k}{k!}\vec{F}^{(k)}(t)$$

Then

$$G'(t) = \frac{(1-t)^n}{n!}\vec{F}^{(n+1)}(t)$$

Proof:

$$G'(t) = \left(\sum_{k=0}^{n} \frac{(1-t)^k}{k!}\vec{F}^{(k)}(t)\right)'$$

$$= \Big[\vec{F}(t) + \dots + \frac{(1-t)^k}{k!}\vec{F}^{(k)}(t) + \frac{(1-t)^{k+1}}{(k+1)!}\vec{F}^{(k+1)}(t) \dots \Big]'$$

$$= \cancel{F'(t)} + \dots - \cancel{\frac{k(1-t)^{k-1}}{k!\,(k-1)!}\vec{F}^{(k)}(t)} + \cancel{\frac{(1-t)^k}{k!}\vec{F}^{(k+1)}(t)}$$

$$- \cancel{\frac{(1-t)^k}{k!}\vec{F}^{(k+1)}(t)} + \cancel{\frac{(1-t)^{k+1}}{(k+1)!}}\vec{F}^{(k+2)}(t)$$

$$+ \frac{(1-t)^n}{n!}\vec{F}^{(n+1)}(t)$$

$$= \frac{(1-t)^n}{n!}\vec{F}^{(n+1)}(t).$$