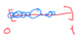
3.6 An alternative description of compactness

Question:  If $\mathcal{O}$ is an infinite collection of open sets such that

$$[0,1] \subseteq \bigcup_{O \in \mathcal{O}} O$$

Is there a finite collection $O_1, O_2, \dots, O_n \in \mathcal{O}$ such that

$$[0,1] \subseteq O_1 \cup O_2 \cup \dots \cup O_n \ ?$$

Answer: Yes! (because $[0,1]$ is compact)

Variant: $(0,1) = \bigcup_{n \in \mathbb{N}} \left(\frac{1}{n}, 1-\frac{1}{n}\right)$

Definition: A subset K of a metric space X has the open covering property (OCP) if whenever $\mathcal{O}$ is a collection of open sets such

$$K \subseteq \bigcup_{O \in \mathcal{O}} O, \quad \leftarrow \text{$\mathcal{O}$ is an open covering of K}$$

then we can find finitely many sets $\underbrace{O_1, O_2, \dots, O_n}_{\text{finite subcovering.}}$ in $\mathcal{O}$ such that

$$K \subseteq O_1 \cup O_2 \cup \dots \cup O_n.$$

AIM K is compact $\Leftrightarrow$ K have the OCP

Proposition: If K has the OCP, then K is compact.

Proof (contrapositive) Assume that K is not compact; we shall construct a covering $\mathcal{O}$ that does not have a finite subcovering.

Since K is not compact, there is a sequence $\{x_n\}$ in K which does not have any subsequence converging to a point in K. If $a \in K$, this means that there is a ball $B(a; r_a)$ around a that only contains finitely many terms of the sequence. The collection

$$\mathcal{O} = \{B(a; r_a)\}_{a \in K}$$

is an open covering of K, but it does not have a finite subcovering. Why

$$\underbrace{B(a_1, r_{a_1}), B(a_2, r_{a_2}), \dots, B(a_n, r_{a_n})}$$

can only contain finitely ~~many~~ many elements of the sequence, and hence there are elements $\underbrace{\text{from the sequence}}_{\text{from } K.}$ that are lacking

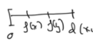
Lemma: Assume that $\mathcal{O}$ is an open covering of a set K. Define $f: K \to [0, \infty)$ by

$$f(x) = \sup\{r \leq 1 : B(x; r) \subseteq O \text{ for some } O \in \mathcal{O}\}.$$

Then f is positive and continuous.

Proof: $f(x) > 0$ because there are $O \in \mathcal{O}$ such that $x \in O$ and since O is open, there is an $r > 0$ such that $B(x; r) \subseteq O$.

To prove that f is continuous, it suffices to show that $|f(x) - f(y)| \leq d(x,y)$. If $f(x), f(y) \leq d(x,y)$, this obvious. Assume



now that at least one of $f(x), f(y)$ is larger that $d(x,y)$, and assume that $f(x) \geq f(y)$. Since $f(x) > d(x,y)$, I will for any r, $d(x,y) < r < f(x)$ have the situation in the figure, and hence the ball

$$\underbrace{B(y, r - d(x,y))}_{\cap\!\!|\ \ O} \subseteq \underbrace{B(x; r)}_{\cap\!\!|\ \ O}$$

$$f(y) \geq f(x) - d(x,y)$$

$$\Downarrow$$

$$d(x,y) \geq |f(x) - f(y)|$$

Theorem: K is compact if and only if it has the OCP.

Proof: It remains to prove that if K is compact, then it has the OCP. So assume that K is compact, and that $\mathcal{O}$ is an open covering of K. We know from the lemma that the function

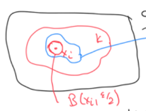
$$f(x) = \sup\{r < 1 : B(x; r) \subseteq O \text{ for some } O \in \mathcal{O}\}.$$

is strictly positive and continuous. By the extremal value theorem, there is an $\varepsilon > 0$ such that $f(x) \geq \varepsilon$ for all $x \in K$.

Since K is compact, it is totally bounded, and hence there is a finite number of ball

$$B(x_1, \varepsilon/2), B(x_2, \varepsilon/2), \dots, B(x_n, \varepsilon/2)$$

that cover K. Since $\varepsilon/2 < f(x_i)$, there is



an O_i in $\mathcal{O}$ such that $B(x_i, \varepsilon/2) \subseteq O_i$. But $\varepsilon/2 \leq f(x_i)$ then

$$K \subseteq B(x_1, \varepsilon/2) \cup \dots \cup B(x_n, \varepsilon/2)$$

$$\subseteq \underbrace{O_1 \cup O_2 \cup \dots \cup O_n}_{\text{finite subcovering.}}$$