Weierstrass' Approximation Theorem

WAT: The polynomial are dense in $C([a,b],\mathbb{R})$; i.e. if $f:[a,b]\to\mathbb{R}$ is continuous, there is a sequence $\{p_n\}$ converging uniformly to f on $[a,b]$.

Theme: Taylor polynomial
$$T_n f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

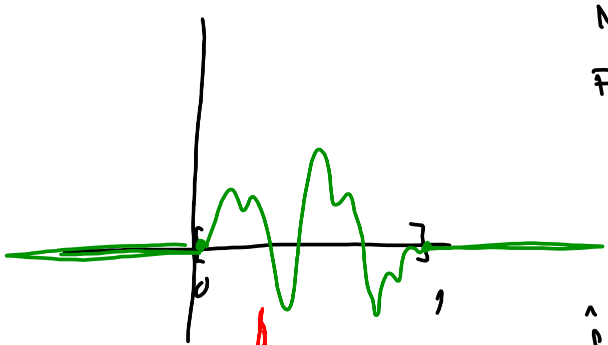
Step 1: For $[0,1]$ with functions f such that $f(0)=f(1)=0$.

Step 2: Remove the condition $f(0)=f(1)=0$.

Step 3: Extend to general intervals $[a,b]$

Step 1

Assume $f:[0,1]\to\mathbb{R}$ is continuous with $f(0)=f(1)=0$. Extend f to a function from $\mathbb{R}$ to $\mathbb{R}$ by putting $f(x)=0$ when $x\notin[0,1]$

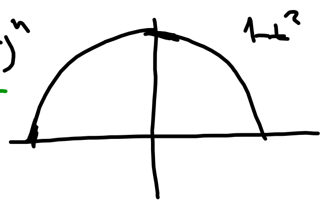


Note that f is uniformly continuous

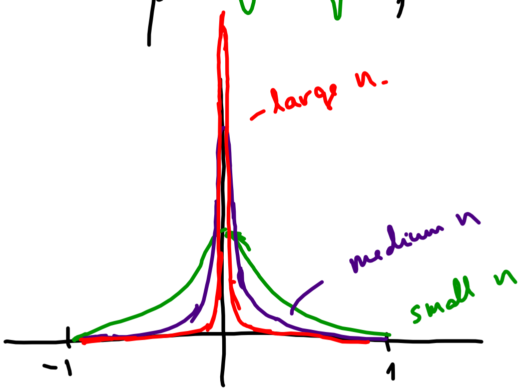
For $n\in\mathbb{N}$, let

$$P_n(t) = c_n(1-t^2)^n$$

where c_n is chosen such that



$$\int_{-1}^{1} P_n(t)\,dt = 1$$



Approximating polynomials

$$p_n(x) = \int_{-1}^{1} f(x+t)P_n(t)\,dt$$

Lemma 1: $p_n(x)$ is a polynomial on $[0,1]$

Proof: $p_n(x) = \int_{-1}^{1} f(x+t)\, P_n(t)\, dt$

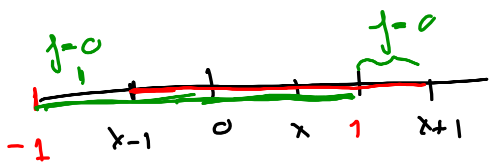
New variable: $y = x+t$

$t = y - x$

$dy = dt$

$$= \int_{x-1}^{x+1} f(y)\, P_n(y-x)\, dy$$

Because $x \in [0,1]$, and thus



$$= \int_{-1}^{1} f(y)\, P_n(y-x)\, dy$$

$$= \int_{-1}^{1} f(y)\, c_n \left(1-(y-x)^2\right)^n dy$$

$$= \int_{-1}^{1} c_n f(y) \sum_{k=0}^{2n} x^k d_k(y)\, dy = \sum_{n=0}^{2n} x^k \underbrace{\int_{-1}^{1} c_n f(y)\, d_k(y)\, dy}_{a_k}$$

$$= \sum_{n=0}^{2n} a_k x^k \leftarrow \text{polynomial.}$$

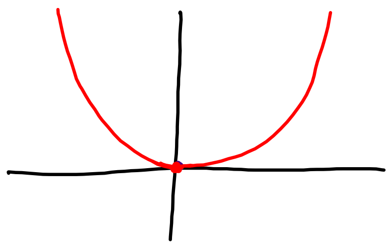
Lemma: For $t \in [-1,1]$, we have

$$(1-t^2)^n \geq 1 - nt^2 .$$

Proof: If $\varphi(t) = (1-t^2)^n - (1-nt^2)$, then $\varphi(0) = 0$.

$$\varphi'(t) = n(1-t^2)^{n-1}(-2t) + 2nt = 2nt - 2nt(1-t^2)^{n-1}$$

$$= 2nt\,(\underbrace{1-(1-t^2)^{n-1}}_{\text{positive on } [-1,1]}) = \begin{cases} >0 & \text{for } t>0 \\ <0 & \text{for } t<0 \end{cases}$$



Hence $\varphi(t) \geq 0$, i.e. $(1-t^2)^n - (1-n t^2)$.

Lemma 3: $c_n < \sqrt{n}$.

Proof: $1 = \int_{-1}^{1} c_n \underbrace{(1-t^2)^n}_{\geq\, 1-nt^2} dt \geq \int_{-\frac{1}{\sqrt{n}}}^{\frac{1}{\sqrt{n}}} c_n (1-nt^2)\, dt$

symmetry

$$= 2c_n \left[t - \frac{n}{3} t^3 \right]_0^{1/\sqrt{n}} = 2c_n \left[\frac{1}{\sqrt{n}} - \frac{n}{3} \frac{1}{n^{3/2}} \right]$$

$$= 2c_n \underbrace{\left[\frac{1}{\sqrt{n}} - \frac{1}{3}\frac{1}{\sqrt{n}} \right]}_{\frac{2}{3}\frac{1}{\sqrt{n}}} = \frac{4}{3} c_n \frac{1}{\sqrt{n}}$$

Hence $\frac{4}{3} c_n \frac{1}{\sqrt{n}} < 1 \Rightarrow c_n \leq \frac{3}{4}\sqrt{n} < \sqrt{n}$.

Proposition: $p_n(x)$ converges uniformly to $f(x)$ on $[0,1]$.

Proof: Given $\varepsilon > 0$, there is an N such that for $n \geq N$, we have

$$|p_n(x) - f(x)| < \varepsilon \text{ for all } x \in [0,1]$$

We have

$$|p_n(x) - f(x)| = \left| \int_{-1}^{1} f(x+t) P_n(t)\,dt - f(x) \underbrace{\int_{-1}^{1} P_n(t)\,dt}_{1} \right|$$

$$\leq \int_{-1}^{1} |f(x+t) - f(x)| P_n(t)\,dt$$

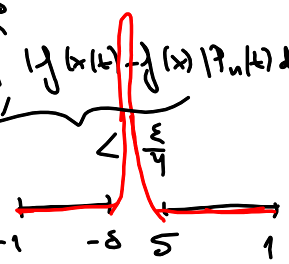
Since f is uniformly continuous, there is a $\delta > 0$ such that if $|t| < \delta$, then $|f(x+t) - f(x)| < \frac{\varepsilon}{4}$. Hence

$$|p_n(x) - f(x)| = \left| \int_{-\delta}^{\delta} \overbrace{|f(x+t) - f(x)|}^{\leq \frac{\varepsilon}{4}} P_n(t)\,dt + \int_{-1}^{\delta} |f(x+t) - f(x)| P_n(t)\,dt \right.$$

$$+ \int_{\delta}^{1} |f(x+t) - f(x)| P_n(t)\,dt$$

$$\leq \int_{-1}^{1} \frac{\varepsilon}{4} |P_n(t)|\,dt + \int_{\delta}^{1} |f(x+t) - f(x)| P_n(t)\,dt + \int_{-1}^{-\delta} |f(x(+t)) - f(x)| P_n(t)\,dt$$

$$= \frac{\varepsilon}{4} + \underbrace{\int_{\delta}^{1} |f(x+t) - f(x)| P_n(t)\,dt}_{< \varepsilon/4} + \underbrace{\int_{-1}^{-\delta} |f(x(+t)) - f(x)| P_n(t)\,dt}_{< \frac{\varepsilon}{4}} < \varepsilon.$$



Let M be the maximal value of $|f|$.

Then

$$\int_{\delta}^{1} \underbrace{|f(x+t) - f(x)|}_{\leq 2M} P_n(t)\,dt \leq \int_{\delta}^{1} 2M c_n \underbrace{(1-t^2)^n}_{\leq (1-\delta^2)^n}\,dt$$

$$\leq 2M \underbrace{c_n}_{\leq \sqrt{n}} (1-\delta^2)^n \cdot 1 \leq 2M \underbrace{\sqrt{n}}_{\infty} \underbrace{(1-\delta^2)^n}_{0} \to 0$$

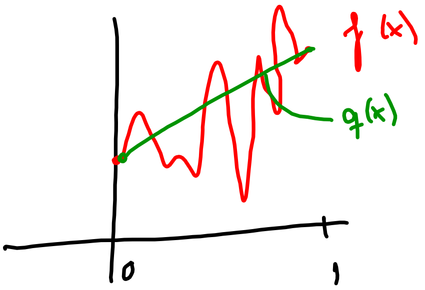
Choosing n large enough we get $\int_{\delta}^{1} |f(x+t) - f(x)| P_n(t) < \frac{\varepsilon}{4}$.

Similarly, we can get $\int_{-1}^{-\delta} |f(x+t) - f(x)| P_n(t)\,dt < \frac{\varepsilon}{4}$ for n large enough.

Step 2:

The result is still true if we drop the requirement that $f(0)=f(1)=0$.

Assume that f is just continuous on $[0,1]$



$g(x) = f(x) - q(x)$, $g(0)=g(1)=0$

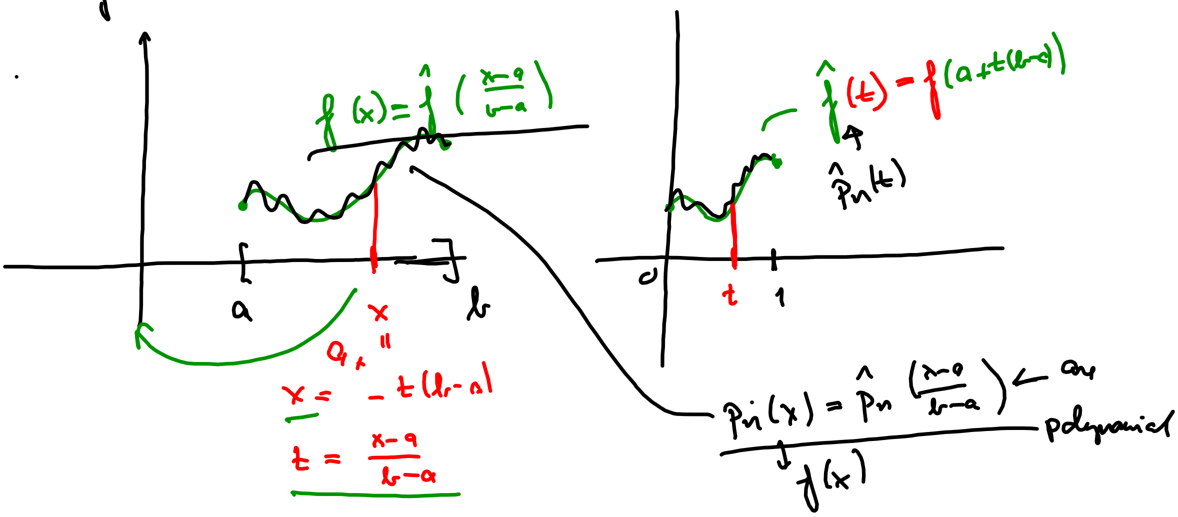
By step 1, there is a sequence $\{p_n\}$ of polynomials s.t. $p_n \to g$ (uniformly)

$$\underbrace{p_n + q}_{\text{polynomial}} \to \underbrace{g+q}_{f} \quad \text{(uniformly)}$$

Step 3

Extending to a general interval $[a,b]$, $a<b$.

$f : [a,b] \to \mathbb{R}$



Since $\hat{p}_n(t) \to \hat{f}(t)$ uniformly on $[0,1]$, then

$p_n(x) \to f(x)$ —— " —— $[a,b]$ and $p_n(x)$ is a polynomial

Normed spaces

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Recall that a <u>vector space</u> / <u>linear space</u> is a set V where we can add elements

$\vec{u}, \vec{v} \in V \qquad \vec{u}+\vec{v}$

and multiply them by scalars

$\alpha \in \mathbb{K}, \vec{u} \in V \qquad \alpha\vec{u}$

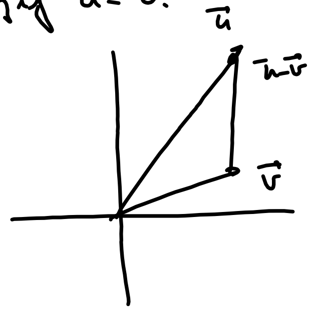
according to "reasonable rules".

<u>Definition</u>. A <u>norm</u> $\|\cdot\|$ on V is a function $\|\cdot\|: V \to \mathbb{R}$ such that:

(i) $\|\vec{u}\| \geq 0$ with equality if and only if $\vec{u}=\vec{0}$.

(ii) $\|\alpha\vec{u}\| = |\alpha|\,\|\vec{u}\|$

(iii) $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\|+\|\vec{v}\|$



<u>Prop</u>: If $\|\cdot\|$ is a norm, then

$$d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$$

defines a metric on V.

<u>Proof</u>: Need to check:

(i) <u>Positivity</u>: $d(\vec{u},\vec{v}) = 0 \Leftrightarrow \|\vec{u}-\vec{v}\| = 0 \Leftrightarrow \vec{u}-\vec{v} = \vec{0}$
$\Leftrightarrow \vec{u}=\vec{v}$.

(ii) <u>Symmetry</u>: $\underline{d(\vec{u},\vec{v})} = \|\vec{u}-\vec{v}\| = \|(-1)(\vec{v}-\vec{u})\| = \underbrace{|-1|}_{1}\,\|\vec{v}-\vec{u}\|$
$= \|\vec{v}-\vec{u}\| = \underline{d(\vec{v},\vec{u})}$.

(iii) <u>Triangle inequality</u>: $d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\| = \|(\vec{u}-\vec{z}) + (\vec{z}-\vec{v})\|$
$\leq \|\vec{u}-\vec{z}\| + \|\vec{z}-\vec{v}\| = d(\vec{u},\vec{z}) + d(\vec{z},\vec{v})$.