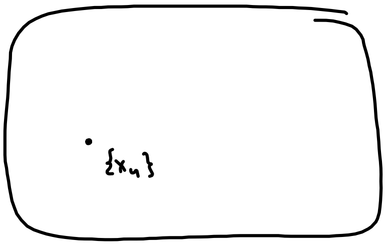
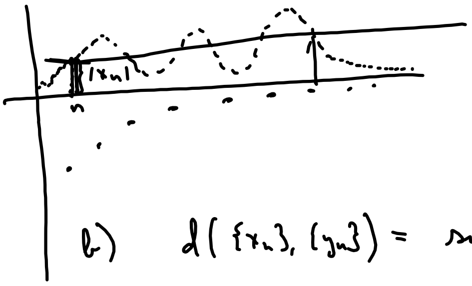
1a) X : $\lim_{n\to\infty} x_n = 0$



$|x_k| = \sup\{|x_n| : n\in\mathbb{N}\}$

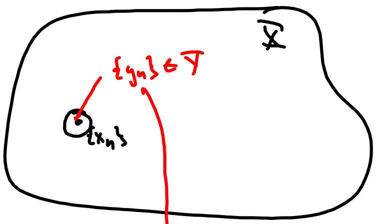


$\{-\frac{1}{n}\}$

b) $d(\{x_n\},\{y_n\}) = \sup\{|x_n - y_n| : n\in\mathbb{N}\}$

(i)
(ii)
(iii)

d)



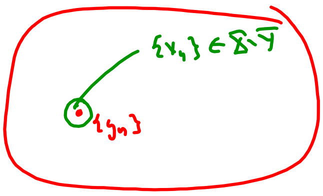
$\{\underset{\sim}{x}\} \in X - Y$

$\lim_{n\to\infty} x_n = 0$

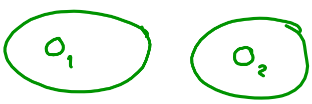
$\sum_{n=1}^{\infty} |x_n| = \infty$

$\sum_{n=1}^{\infty} |y_n| < \infty$

e)



2 Disconnected



a) $X = \underbrace{[0,1]}_{O_1} \cup \underbrace{[2,3]}_{O_2}$ $d(x,y) = |x-y|$

$O_1 \cup O_2 = X$

$O_1 \cap O_2 = \emptyset$

b)

Section 4.2: Uniform convergence

Def: Let (X, d_X) and (Y, d_Y) be two metric spaces and let $\{f_n\}$ be a sequence of functions $f_n : X \to Y$. We say that $\{f_n\}$ converges pointwise to a function $f : X \to Y$ if for all $x \in X$, the sequences $\{f_n(x)\} \to f(x)$

Example: $f_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$

$f_n(x) \to e^x$

This means that for each x and each $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that if $n \geq N$, then $d(f_n(x), f(x)) < \varepsilon$

$N(\varepsilon, x)$

Def: We say $\{f_n\}$ converges uniformly to f if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that when $n \geq N$, then

$$d(f_n(x), f(x)) < \varepsilon \quad \text{for all } x$$

$N(\varepsilon)$

Proposition: The following are equivalent

(i) $f_n \to f$ uniformly

(ii) $\sup\{|f_n(x) - f(x)| : x \in X\} \to 0$ as $n \to \infty$.

Proof: (i) $\Rightarrow$ (ii) Given an $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for $n \geq N$ and $x \in X$

$$|f_n(x) - f(x)| < \varepsilon$$

But then

$$\sup\{|f_n(x) - f(x)| : x \in X\} \leq \varepsilon$$

which means that

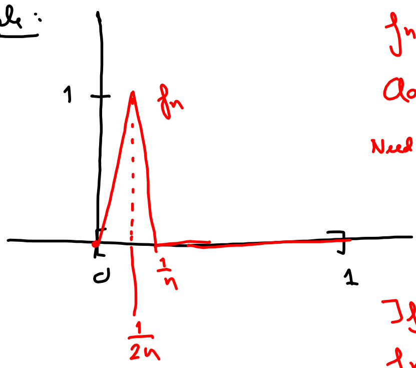
$$\sup\{|f_n(x) - f(x)| : x \in X\} \to 0$$

(ii) $\Rightarrow$ (i) Assume (ii), then for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for $n \geq N$,

$$\sup\{|f_n(x) - f(x)| : x \in X\} < \varepsilon$$

But then $|f_n(x) - f(x)| < \varepsilon$ for all $x \in X$ and all $n \geq N$, and hence f_n converges uniformly to f.

Example:



$f_n : [0,1] \to \mathbb{R}$

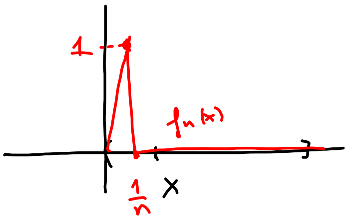
Claim: $f_n(x) \to 0$ pointwise as $n \to \infty$

Need to prove

$$\lim_{n\to\infty} f_n(x) = 0 \quad \text{for } x \in [0,1]$$

OK for $x = 0$ (because $f_n(0) = 0$)

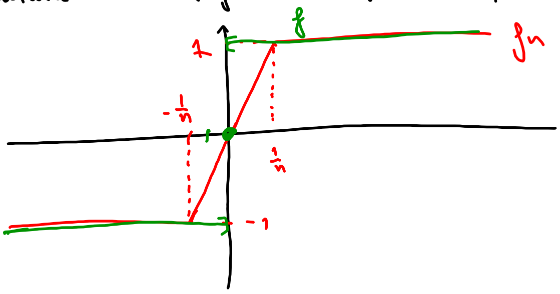
If $x > 0$, then eventually $f_n(x) = 0$ because $\frac{1}{n} < x$.



$\sup\{|f_n(x) - f(x)| : x \in [0,1]\} = 1$ ($f = 0$)

$\not\to 0$ not uniform convergence.

Example: If f_n are continuous and $f_n \to f$, will f be continuous? No, if the convergence is only pointwise:



The f_n's are continuous

Theorem: Assume that a sequence $\{f_n\}$ of continuous functions $f_n : X \to Y$ converges uniformly to f. Then f is a continuous.

Proof: Let $a \in X$. To prove that f is continuous at a we have to show that given an $\varepsilon > 0$ there is a $\delta > 0$ such

$$d_X(a,x) < \delta \Rightarrow d_Y(f(a), f(x)) < \varepsilon.$$

Given $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that if $n \geq N$, then $d_Y(f_N(x), f(x)) < \frac{\varepsilon}{3}$ for all $x \in X$.

Since f_N is continuous, there is a $\delta > 0$ such that if $d_X(a,x) < \delta$ then $d_Y(f_N(a), f_N(x)) < \frac{\varepsilon}{3}$.

But then

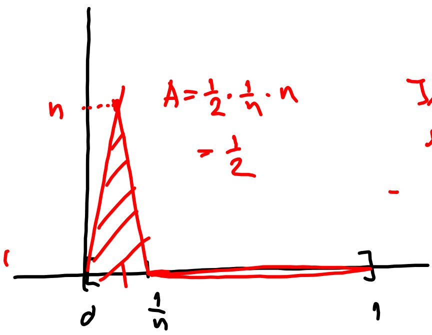
$$d_Y(f(a), f(x)) \leq \underbrace{d_Y(f(a), f_N(a))}_{<\varepsilon/3} + \underbrace{d_Y(f_N(a), f_N(x))}_{<\varepsilon/3} + \underbrace{d_Y(f_N(x), f(x))}_{<\varepsilon/3}$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

4.3 Integration of sequences

Question: If $f_n \to f$, will
$$\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx \;?$$

Answer: No, if you mean pointwise convergence



$A = \frac{1}{2} \cdot \frac{1}{n} \cdot n = \frac{1}{2}$

$f_n(x) \to 0$ pointwise

If the assumption were true, then
$$\int_0^1 f_n(x)\,dx \to \int_0^1 0\,dx = 0$$
But they don't
$$\int_0^1 f_n(x)\,dx = \frac{1}{2}.$$

Theorem: Assume that $f_n : [a,b] \to \mathbb{R}$ is a sequence of continuous functions converging uniformly to f. Then the function $F_n(x) = \int_a^x f_n(t)\,dt$ will converge uniformly to $F(x) = \int_a^x f(t)\,dt$ on $[a,b]$. In particular,
$$\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx.$$

Proof: Since $f_n \to f$ uniformly, there is for each $\varepsilon > 0$ an $N \in \mathbb{N}$ such that for $n \geq N$ we have $|f_n(t) - f(t)| < \frac{\varepsilon}{b-a}$ for all $t \in [a,b]$. But then
$$|F_n(x) - F(x)| = \left| \int_a^x f_n(t)\,dt - \int_a^x f(t)\,dt \right|$$
$$= \left| \int_a^x (f_n(t) - f(t))\,dt \right| \leq \int_a^x |f_n(t) - f(t)|\,dt < \int_a^x \frac{\varepsilon}{b-a}\,dt$$
$$= \frac{\varepsilon}{b-a} \underbrace{(x-a)}_{\leq\, b-a} \leq \varepsilon$$
Hence F_n converges uniformly to F on $[a,b]$.

Question: $f_n \to f$, $f_n' \to f'$?

$$\frac{\sin nx}{n} \to 0$$