Normed spaces

V is a linear space. $\|\;\|: V \to \mathbb{R}$ is a <u>norm</u> on V if

(i) $\|\vec{u}\| \geq 0$ with equality iff $\vec{u} = \vec{0}$

(ii) $\|\alpha \vec{u}\| = |\alpha| \|\vec{u}\|$

(iii) $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

Associated metric: $d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$

$$\vec{x}_n \to \vec{x} \iff \underline{d(\vec{x}_n,\vec{x}) \to 0} \iff \underline{\|\vec{x}_n - \vec{x}\| \to 0}$$

Examples: (i) $\mathbb{R}^n$, $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

(ii) $\mathbb{C}^n$, $\|\vec{z}\| = \sqrt{|z_1|^2 + \dots + |z_n|^2}$

(iii) $C([a,b],\mathbb{R})$

$$\|f\|_\infty = \sup\{|f(t)| : t \in [a,b]\}$$

$$\|f\|_1 = \int_a^b |f(t)|\,dt$$

$$\|f\|_2 = \left(\int_a^b |f(t)|^2\,dt\right)^{1/2}$$

Inverse Triangle Inequality: $\big|\|\vec{u}\| - \|\vec{v}\|\big| \leq \|\vec{u}-\vec{v}\|$

Proof: $\big|\underbrace{\|\vec{u}\|}_{\|\vec{u}-\vec{0}\|} - \underbrace{\|\vec{v}\|}_{\|\vec{v}-0\|}\big| = |d(\vec{u},\vec{0}) - d(\vec{v},\vec{0})| \leq d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$

ITI for metric space.

Proposition:

(i) If $\{\vec{x}_n\}$ converges to $\vec{x}$, then $\{\|\vec{x}_n\|\}$ converges to $\|\vec{x}\|$.

(ii) If $\{\vec{x}_n\}$ converges to $\vec{x}$ and $\{\vec{y}_n\}$ converges to $\vec{y}$, then $\{\vec{x}_n + \vec{y}_n\}$ converges to $\vec{x}+\vec{y}$.

(iii) If $\{\alpha_n\}$ converges to α and $\{\vec{x}_n\}$ converges to $\vec{x}$, then

$$\alpha_n x_n \to \alpha x.$$

Proof (i) We have

$$0 \leq \big|\underbrace{\|\vec{x}_n\| - \|\vec{x}\|}_{\to 0}\big| \leq \|\vec{x}_n - \vec{x}\| \to 0$$

and hence $\|\vec{x}_n\| \to \|\vec{x}\|$.

(iii) $0 \leq \|\alpha_n \vec{x}_n - \alpha\vec{x}\| \leq \|\underbrace{\alpha_n \vec{x}_n - \alpha\vec{x}_n} + \underbrace{\alpha\vec{x}_n - \alpha\vec{x}}\|$

$$\leq \|\alpha_n \vec{x}_n - \alpha\vec{x}_n\| + \|\alpha\vec{x}_n - \alpha\vec{x}\|$$

$$= \|(\alpha_n - \alpha)\vec{x}_n\| + \|\alpha(\vec{x}_n - \vec{x})\|$$

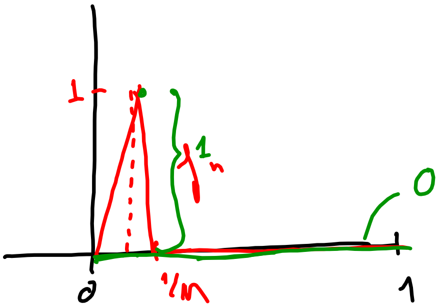
$$= \underbrace{|\alpha_n - \alpha|}_{\to 0}\underbrace{\|\vec{x}_n\|}_{\to \|\vec{x}\|} + \underbrace{|\alpha|}_{\to |\alpha|}\underbrace{\|\vec{x}_n - \vec{x}\|}_{\to 0} \to 0.$$

Hence $\|\alpha_n \vec{x}_n - \alpha\vec{x}\| \to 0$, i.e. $\alpha_n\vec{x}_n \to \alpha\vec{x}$

Example: $V = C([0,1], \mathbb{R})$. Two norms

$$\|f\|_\infty = \sup\{|f(t)| : t \in [0,1]\}$$

$$\|f\|_1 = \int_0^1 |f(t)|\,dt$$



Claim that $f_n \to 0$ in $\|\ \|_1$-norm:

$$\|f_n - 0\|_1 = \int_0^1 |f_n(t) - 0|\,dt$$

$$= \int_0^1 |f_n(t)|\,dt = \frac{1}{2}\cdot\frac{1}{n}\cdot 1 = \frac{1}{2n} \to 0$$

Claim that $f_n \not\to 0$ in $\|\ \|_\infty$-norm.

$$\|f_n - 0\|_\infty = 1 \not\to 0 \text{ as } n \to \infty.$$

Definition: Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a linear space V are equivalent if there are constants K_1 and K_2 such that

$$\|\vec{v}\|_1 \leq K_1 \|\vec{v}\|_2 \quad \text{for all } \vec{v} \in V$$

$$\|\vec{v}\|_2 \leq K_2 \|\vec{v}\|_1 \quad \text{for all } \vec{v} \in V.$$

Proposition: Assume that $\|\cdot\|_1$ and $\|\ \|_2$ are two equivalent norms on V.

(i) A sequence $\{\vec{x}_n\}$ converges to $\vec{x}$ in the $\|\cdot\|_1$-norm if and only if it converges in $\|\cdot\|_2$-norm

(ii) A subset $A \subseteq V$ is open/closed/compact with respect to $\|\cdot\|_1$ if and only if it is open/closed/compact with respect to $\|\cdot\|_2$.

(iii) A function $f: V \to \underbrace{X}_{\text{metric}}$ or $g: X \to V$ is continuous with respect to $\|\cdot\|_1$ if and only if it is continuous with respect to $\|\cdot\|_2$.

Theorem: All norms on $\mathbb{R}^n$ are equivalent.

Remark: Note that if we have three norms $|\ |_1 \sim |\ |_2$ and $|\ |_2 \sim |\ |_3$ then $|\ |_1 \sim |\ |_3$. Hence it is sufficient to show that all norms on $\mathbb{R}^n$ are equivalent to

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}.$$

Proof: Assume that $|\cdot|$ is a norm on $\mathbb{R}^n$ and that $\|\cdot\|$ is the euclidean norm on $\mathbb{R}^2$. We need to show that there are constants k_1 and k_2 such that

$$|\vec{u}| \leq k_1 \|\vec{u}\| \text{ for all } \vec{u} \in \mathbb{R}^n$$

$$\|\vec{u}\| \leq k_2 |\vec{u}| \text{ for all } \vec{u} \in \mathbb{R}^n$$

Any $\vec{u} \in \mathbb{R}^n$ can be written $\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 + \dots + u_n\vec{e}_n$.

Let B be the largest of the numbers $|\vec{e}_1|, \dots, |\vec{e}_n|$. Then

$$|\vec{u}| = |u_1\vec{e}_1 + u_2\vec{e}_2 + \dots + u_n\vec{e}_n| \leq |u_1\vec{e}_1| + |u_2\vec{e}_2| + \dots + |u_n\vec{e}_n|$$

$$= |u_1||\vec{e}_1| + |u_2||\vec{e}_2| + \dots + |u_n||\vec{e}_n| \leq nB\max\{|u_i| : i = 1,\dots,n\}$$

$$\leq nB\|\vec{u}\| \quad \text{because } u_i \leq \sqrt{u_1^2 + \dots + u_i^2 + \dots + u_n^2} = \|\vec{u}\|$$

If we put $k_1 = nB$, we have

$$|\vec{u}| \leq k_1\|\vec{u}\|.$$

To prove the opposite inequality, define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(\vec{u}) = |\vec{u}|$. This function is continuous since

$$|f(\vec{u}) - f(\vec{v})| = ||\vec{u}| - |\vec{v}|| \leq |\vec{u} - \vec{v}| \leq k\|\vec{u} - \vec{v}\|$$

By what we have just proved.

The set

$$K = \{\vec{u} \in \mathbb{R}^n : \|\vec{u}\| = 1\}$$

is closed and bounded (i.e. compact) in $\mathbb{R}^n$ with $\|\cdot\|$.

By the Extremal value Theorem, f has a minimum point $\vec{u}_0$ on K. Hence for $\vec{u} \in K$,

$$f(\vec{u}) \geq f(\vec{u}_0) = |\vec{u}_0| = \alpha > 0. \quad \text{since } \vec{u}_0 \neq \vec{0}.$$

For any $\vec{u} \in \mathbb{R}^n$, $\frac{\vec{u}}{\|\vec{u}\|} \in K$, and hence

$$\frac{1}{\|\vec{u}\|}|\vec{u}| = \left|\frac{\vec{u}}{\|\vec{u}\|}\right| = f\left(\frac{\vec{u}}{\|\vec{u}\|}\right) \geq \alpha$$

$\frac{1}{\alpha}|\vec{u}| \geq \|\vec{u}\|$. Put $k_2 = \frac{1}{\alpha}$, and we get

$$\|\vec{u}\| \leq k_2|\vec{u}| \qquad \text{QED.}$$

Series and bases

Linear algebra: A basis for V is a set $\vec{e}_0, \dots, \vec{e}_n$ such that any element $\vec{v} \in V$ can be written as a linear combination

$$\vec{v} = \alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2 + \dots + \alpha_n \vec{e}_n \quad n \text{ is the dimension of } V.$$

Need to extend this notion to infinite bases $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n, \dots$

Have to make sense of infinite sums $\sum_{n=0}^{\infty} \alpha_n \vec{e}_n$

Assume that we have an infinite sequence of element $\{\vec{v}_n\}$ and we want to define the infinite sum $\sum_{n=0}^{\infty} \vec{v}_n$.

$\vec{s}_0 = \vec{v}_0$, $\vec{s}_1 = \vec{v}_0 + \vec{v}_1$, $\vec{s}_2 = \vec{v}_0 + \vec{v}_1 + \vec{v}_2, \dots$

Partial sums $\vec{s}_n = \sum_{k=1}^{n} \vec{v}_k$.

Definition. We say that the series $\sum_{n=0}^{\infty} \vec{v}_n$ converges if the partial sums s_n converge to an element $\vec{s} \in V$, and we define the sum by $\sum_{n=0}^{\infty} \vec{v}_n = s$

$\sum a_n = \sum b_n$

Definition: A set $\{\vec{e}_0, \vec{e}_1, \dots, \vec{e}_n, \dots\}$ of elements in V is a basis for V if every element $\vec{v} \in V$ can be written as a sum

$$\vec{v} = \sum_{i=0}^{\infty} \alpha_n \vec{e}_n$$

in a unique way

$$\sum_{n=0}^{\infty} \alpha_n \vec{e}_n - \sum_{n=0}^{\infty} \beta_n \vec{e}_n$$

$$\Downarrow$$

$\alpha_n = \beta_n$ for all n.

Warning: $\vec{u} = \sum_{n=0}^{\infty} \vec{v}_n$ looks like algebra, but depends on what norm we are using:

$$\|\vec{u} - \sum_{n=0}^{N} \vec{v}_n\|_1 \to 0$$

$$\|\vec{u} - \sum_{n=0}^{N} \vec{v}_n\|_2 \overset{?}{\not\to} 0$$