What is real analysis? A continuation of MAT 1100
MAT 1110, MAT 1120

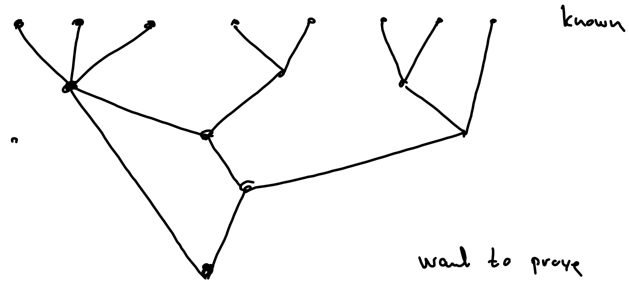
Main concepts: The 4C's: continuity, convergence, completeness, compactness (not calculations)

integrals, derivatives, limits, series, norm, inner products

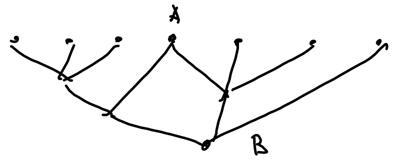
Rumor: MAT 2400 is hard!

New:
- abstract (vector spaces) ←
- proofs (nothing new) ←

Proofs



Direct proofs: If A, then B ($A \Rightarrow B$)



Example: If x and y are rational numbers, then $x+y$ is a rational number.

Proof: Since x and y are rational

$$x = \frac{a}{b}, \quad y = \frac{c}{d} \quad \text{where } a,b,c,d \text{ are integers}$$

$$x+y = \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}$$ integers

and hence $x+y$ is rational.

Contrapositive proofs: Want to prove that $A \Rightarrow B$. Instead I prove that $\neg B \Rightarrow \neg A$, and then I will have proved $A \Rightarrow B$.

$\neg B$: not-B, $\neg A$: not A.

(Why? Assume we know that $\neg B \Rightarrow \neg A$ holds. If then A holds, I can not possibly have $\neg B$, because then we would also have $\neg A$, which is impossible.)

Example: If n is a natural number and n^2 is even, then n is even.

Proof:

$$\underbrace{n^2 \text{ is even}}_{A} \Rightarrow \underbrace{n \text{ is even}}_{B}$$

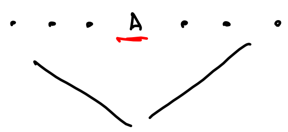
$$\underbrace{n \text{ is odd}}_{\neg B} \Rightarrow \underbrace{n^2 \text{ is odd}}_{\neg A}$$

Assume that n is odd: $n = 2u+1$

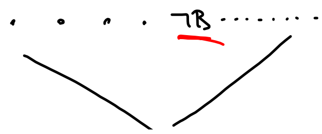
Then $n^2 = (2u+1)^2 = 4u^2 + 4u + 1 = \underbrace{2(2u^2+2u)+1}_{\text{odd}}$

Compare:

Direct proof

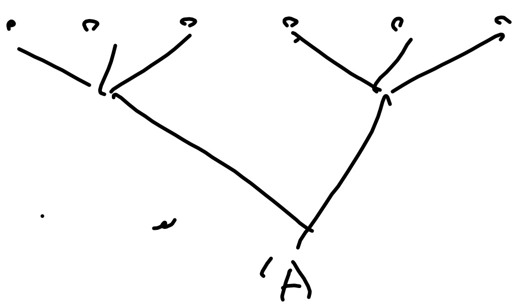


Contrapositive proof

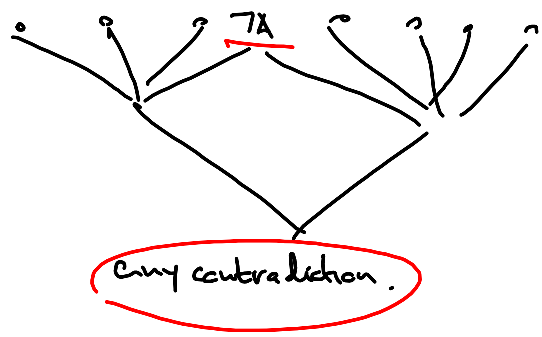


Proof by contradiction: Want to prove A. Assume $\neg A$ and show that this leads to a contradiction.

Direct proof.



Proof by contradiction:



Example: Show that if x is rational and y is irrational, then $x+y$ is irrational.

Proof: Assume for contradiction that $x+y=z$ is rational. Then $y = \underbrace{z}_{\text{rational}} - \underbrace{x}_{\text{rational}}$ is rational because it's the difference numbers

This is a contradiction since y was assumed to be irrational.

Sets

A set is a collection of mathematical objects:

$A = \{1,2,3,\dots,10\}$

$[a,b] = \{x \in \mathbb{R} : a \le x \le b\}$

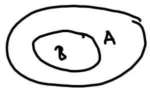
$= \{x \in \mathbb{R} \mid a \le x \le b\}$

Notation: $x \in A$ "x is an element/member of the set A"

$x \notin A$ " —— is not —— " ——"

$2 \in A$, $11 \notin A$.

Subset: $B \subseteq A$ "all the elements in B are elements in A"



$\{1,2,3\} \subseteq A$

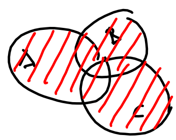
$\{1,2,11\} \nsubseteq A$

$2 \nsubseteq A$

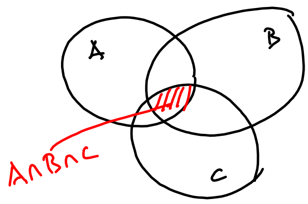
$A = B$ iff $A \subseteq B$ and $B \subseteq A$

Operations with sets

Union: $x \in A_1 \cup A_2 \cup \dots \cup A_n \Leftrightarrow$ x belongs to at least one of the sets $A_1, A_2, \dots, A_n$

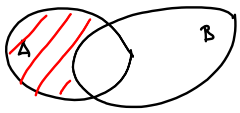


Intersection. $x \in A_1 \cap A_2 \cap \dots \cap A_n \Leftrightarrow$ x belongs to all the sets $A_1, A_2, \dots, A_n$



Set-theoretic difference:

$A \setminus B = \{x \in A : x \notin B\}$



Distributive Laws:

$x(y_1 + y_2 + \dots + y_n) = xy_1 + xy_2 + \dots + xy_n$

(i) $B \cap (A_1 \cup A_2 \cup \dots \cup A_n) = (B \cap A_1) \cup (B \cap A_2) \cup \dots \cup (B \cap A_n)$

(ii) $B \cup (A_1 \cap A_2 \cap \dots \cap A_n) = (B \cup A_1) \cap (B \cup A_2) \cap \dots \cap (B \cup A_n)$

Proof (i) I'll first prove $\subseteq$: Assume that

$x \in B \cap (A_1 \cup A_2 \cup \dots \cup A_n) \Rightarrow x \in B$ and $x \in A_1 \cup \dots \cup A_n$

$\Rightarrow x \in B$ and $x \in A_i$ for some i $\Rightarrow x \in B \cap A_i$

$\Rightarrow x \in (B \cap A_1) \cup (B \cap A_2) \cup \dots \cup (B \cap A_n)$.

Next we prove $\supseteq$: Assume that

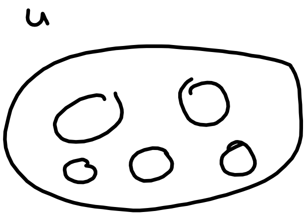
$x \in (B \cap A_1) \cup (B \cap A_2) \cup \dots \cup (B \cap A_n) \Rightarrow x \in B \cap A_i$ for some i

$\Rightarrow x \in B$ and $x \in A_i$ for some $i \Rightarrow x \in B$ and $x \in A_1 \cup \dots \cup A_n$

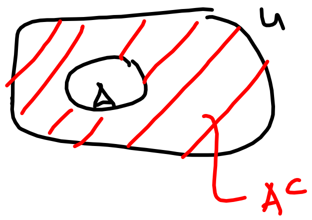
$\Rightarrow x \in B \cap (A_1 \cup A_2 \cup \dots \cup A_n)$

Universe

Often all the sets we are interested in are subsets of a given set or "universe" U. Given a universe U and $A \subseteq U$, then the complement A^c is defined by



$$A^c = U \setminus A = \{x \in U : x \notin A\}$$



De Morgan's Laws:

(i) $(A_1 \cup \dots \cup A_n)^c = A_1^c \cap A_2^c \cap \dots \cap A_n^c$

(ii) $(A_1 \cap \dots \cap A_n)^c = A_1^c \cup A_2^c \cup \dots \cup A_n^c$

Proof: (i) Assume $x \in U$: $x \in (A_1 \cup \dots \cup A_n)^c \iff x \notin A_1 \cup \dots \cup A_n$

$\iff$ For all i: $x \notin A_i \iff$ For all i: $x \in A_i^c$

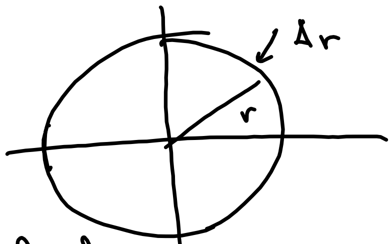
$\iff x \in A_1^c \cap \dots \cap A_n^c$

Family of sets

A family $\mathcal{A}$ of sets is a finite or infinite collection of sets

Example: If $r \geq 0$, let

$$A_r = \{(x,y) : x^2+y^2=r^2\}$$



$$\mathcal{A} = \{A_r\}_{r\geq 0}$$

Definition: Assume that $\mathcal{A}$ is a family of sets. Then

$x \in \bigcup\limits_{A\in\mathcal{A}} A = \bigcup\{A : A\in\mathcal{A}\} \iff x$ belongs to at least one $A \in \mathcal{A}$

$x \in \bigcap\limits_{A\in\mathcal{A}} A = \bigcap\{A : A\in\mathcal{A}\} \iff x$ belongs to all $A \in \mathcal{A}$