Review

Metric spaces

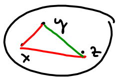
Def: d is a metric on X if $d: X \times X \to \mathbb{R}$ s.t.

(i) $d(x,y) \geq 0$ with equality iff $x = y$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,y) \leq d(x,z) + d(z,y)$

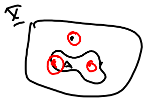
Inverse triangle inequality $|d(x,y) - d(x,z)| \leq d(y,z)$



$\varepsilon-\delta$, $\varepsilon-N$ definitions

Limit: $\{x_n\}$ converges to x if for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N$.

Continuity: $f: X \to Y$ is continuous at $a \in X$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $d_X(x,a) < \delta$, then $d_Y(f(x), f(a)) < \varepsilon$.



Subsets: If $A \subseteq X$, we have

(i) Interior points: There is an $r > 0$ such that $B(a;r) \subseteq A$

(ii) Exterior points: There is an $r > 0$ such that $B(a;r) \subseteq A^c$

(iii) Boundary points: All balls $B(a,r)$ contain points from both A and A^c.

Open set: A set $A \subseteq X$ is open if it doesn't contain any boundary points.

Closed set: A set $A \subseteq X$ is closed if it contains all its boundary points

Other descriptions: TFAE

(i) A is open

(ii) all points in A are interior.

(iii) A^c is closed

Other descriptions: TFAE

(i) A is closed

(ii) A^c is open

(iii) If $\{x_n\}$ is a sequence from A converging to x, then $x \in A$.

Exam 2021, problem 2: $A \subseteq \mathbb{R}$ is closed.

$C_b(X, \mathbb{R}) = \{f: X \to \mathbb{R} : f \text{ cont. and bounded}\}$

Prove that

$C_b(X, A) = \{f: X \to A : f \text{ cont. and bounded}\}$

is closed.

Solution no 1: Assume that $\{f_n\}$ is a sequence from $C_b(X,A)$ converging to f (in $C_b(X,\mathbb{R})$). We must prove that $f \in C_b(X,A)$. That $f_n \to f$ in $C_b(X,\mathbb{R})$ means uniform convergence, which implies pointwise convergence, so $\underset{\in A}{\underbrace{f_n(x)}} \to \underset{\in A}{f(x)}$ for all x. Hence $f \in C_b(X,A)$

$\Rightarrow$ since A is closed.

Solution no 2: It suffices to prove that $C_b(X,A)^c$ is open. Choose $f \in C_b(X,A)^c$, we want to show that f is an interior point, i.e. that there exists an $r > 0$ such that $B(f,a) \subseteq C_b(X,A)^c$.

Since $f \notin C_b(X,A)$, there has to be an $x \in X$ such that $f(x) \notin A$. Since A is closed, A^c is open and hence there is an $r > 0$ such that $B(f(x), r) \subset A^c$. But then $B(f,r)$ in $C_b(X,\mathbb{R})$ does not contain

Alternative descriptions of continuity: TFAE

(i) f is continuous at a

(ii) for all sequences $x_n \to a$, $f(x_n) \to f(a)$

Alternative descriptions: TFAE

(i) f is continuous (at all points)

(ii) $f^{-1}(O)$ is open for all open $O \in Y$.

(iii) $f^{-1}(F)$ is closed for all closed $F \in Y$.

Completeness

Cauchy sequence: $\{x_n\}$ is a a *Cauchy sequence* if for any $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ when $n, m \geq N$.

Obs: All convergent sequences are Cauchy.

Def: X is *complete* if all Cauchy sequences converge.

Examples: (i) $\mathbb{R}^n$ is complete (but not $\mathbb{Q}^n$)

(ii) If Y is complete, then $B(X,Y)$ and $C_b(X,Y)$ are complete in the supremum metric

$$\rho(f,g) = \sup\{d(f(x), g(x)) : x \in X\}$$

Prop: A subset A of a complete space is complete if and only if A is closed.

Banach's Fixed Point Theorem: Assume that X is complete and $f: X \to X$ is a *contraction*. Then f has a fixed point x ($x = f(x)$) and for any $x_0 \in X$, the sequence $x_0, f(x_0), f(f(x_0)), \ldots$ converges to x.

$d(f(x), f(y)) \leq s\, d(x,y)$

$s < 1$.

Compactness

Subsequence: Assume $\{x_n\}$ is a sequence in X.

$$x_1,\ \underset{\substack{\uparrow\\ n_1}}{x_2},\ \underset{\substack{\uparrow\\ n_2}}{x_3},\ x_4,\ \underset{\substack{\uparrow\\ n_3}}{x_5},\ \dots\ \underset{\substack{\uparrow\\ n_4}}{\ }$$

Let $n_1 < n_2 < n_3 < \dots$ be an increasing of integers. Then $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$

Compactness: A subset K of a metric space X is compact if every sequence $\{x_n\}$ of elements from K has a subsequenc $\{x_{n_k}\}$ which converges to a point p in K.

Examples:

(i) Bolzano-Weierstrass: a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

(ii) Arzela-Ascoli: A subset K of $C(X,\mathbb{R}^m)$ (for X compact) is compact if and only if it is closed, bounded and equicontinuous.

Useful fact: If K is compact and f is continuous, then $f(K)$ is compact.

Alternative descriptions: TFAE

(i) A is compact

(ii) all open coverings of A have finite subcoverings.

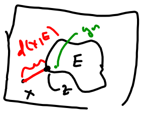
(iii) (assuming X complete) A is closed and totally bounded

Extreme Value Theorem: If X is compact and $f: X \to \mathbb{R}$ is continuous, then f has a maximum point c and a minimum point d on X, i.e.

$$f(d) \le f(x) \le f(c) \quad \text{for all } x \in X.$$

Exam 2021, problem 3: X metric space, $E \subseteq X$, $x \in X$

Define $d(x,E) = \inf\{d(x,y) : y \in E\}$



a) Show that if E is compact, there is a point $z \in E$ such that $d(x,E) = d(x,z)$

Solution no 1: For any $n \in \mathbb{N}$, I can find an element $y_n \in E$ such that $d(x,y_n) \le d(x,E) + \frac{1}{n}$.

This gives us a sequence $\{y_n\}$ from E, and since E is compact, there is a convergent subsequence $\{y_{n_k}\}$ converging to a point $y \in E$. But $d(x,y) = \lim d(x,y_{n_k}) = d(x,E)$

Solution no 2: Define $f(y) = d(x,y)$. This is a continuous function, so by EVT it has a minimum point y on E. Then we must have

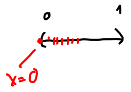
$$f(y) = d(x,y) = d(x,E)$$

(f is continuous because

$|f(y) - f(z)| = |d(x,y) - d(x,z)|$

$\le d(y,z)$ $\quad \delta = \varepsilon$.)

b) A counter example when E is not compact.



$E = (0,1)$

$d(x,E) = 0$

$d(x,y) > 0$ for $y \in E$