Family of sets

A *family* $\mathcal{A}$ is just a collection of sets.

$x \in \bigcup_{A \in \mathcal{A}} A \iff x$ is an element of at least one $A \in \mathcal{A}$

$\forall$
$x \in \bigcap_{A \in \mathcal{A}} A \iff x$ is an element of all $A \in \mathcal{A}$

Distributive laws:

(i) $B \cap \bigcup_{A \in \mathcal{A}} A = \bigcup_{A \in \mathcal{A}} (B \cap A)$

(ii) $B \cup \bigcap_{A \in \mathcal{A}} A = \bigcap_{A \in \mathcal{A}} (B \cup A)$

De Morgan's Laws:

(i) $\left(\bigcup_{A \in \mathcal{A}} A\right)^c = \bigcap_{A \in \mathcal{A}} A^c$

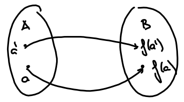
(ii) $\left(\bigcap_{A \in \mathcal{F}} A\right)^c = \bigcup_{A \in \mathcal{F}} A^c$

Proof (ii) Assume $x \in U$. $x \in \left(\bigcap_{A \in \mathcal{A}} A\right)^c \iff x \notin \bigcap_{A \in \mathcal{A}} A \iff$ There is an $A \in \mathcal{A}$ such that $x \notin A$.

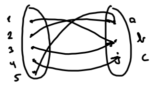
$\iff$ There is an $A \in \mathcal{A}$ such that $x \in A^c \iff x \in \bigcup_{A \in \mathcal{A}} A^c$

Functions (Sec 1.4)

Assume that A and B are two nonempty sets. A function f from A to B (in symbols $f: A \to B$) is an $\underbrace{\text{assignment}}^{\text{rule}}$ that to each $a \in A$ assigns an element $f(a)$ in B

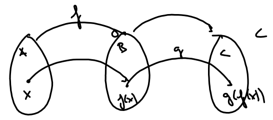


Example $A = \{1,2,3,4,5\}$, $B = \{a, b, c\}$



$f(1) = b$ $f(2) = a$ $f(3) = b$
$f(4) = c$ $f(5) = a$.

Definition: Assume $f: A \to B$ and $g: B \to C$.

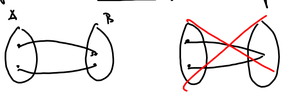


The composition $g \circ f$ is the function from A to C given by

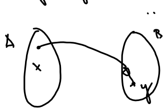
$$(g \circ f)(x) = g(f(x))$$

Properties of functions

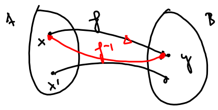
Definition: $f: A \to B$ is injective if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$



Definition: $f: A \to B$ is surjective if for all $y \in B$ there is an $x \in A$ such that $f(x) = y$



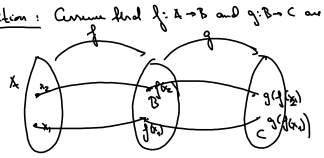
Definition: $f: A \to B$ is bijective if it is both injective and surjective. This means that for every $y \in B$ there is exactly one $x \in A$ such that $y = f(x)$.



If $f: A \to B$ is bijective, we define the inverse function $f^{-1}: B \to A$ by saying that for each $y \in B$, $f^{-1}(y)$ is the unique element $x \in A$ such that $f(x) = y$.

Note: $f^{-1}(f(x)) = x$ $\quad f(f^{-1}(y)) = y$

Proposition: Assume that $f: A \to B$ and $g: B \to C$ are bijective



Then $g \circ f$ is also bijective and

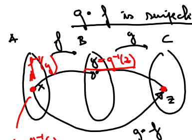
$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

Proof: $g \circ f$ is injective: Assume $x_1 \neq x_2$, need to show $g \circ f(x_1) \neq g \circ f(x_2)$

Since f is injective, $f(x_1) \neq f(x_2)$, and hence $\underbrace{g(f(x_1))}_{g \circ f(x_1)} \neq \underbrace{g(f(x_2))}_{g \circ f(x_2)}$ since g is injective. This means that $g \circ f(x_1) \neq g \circ f(x_2)$ and hence $g \circ f$ is injective

$g \circ f$ is surjective: Since g is surjective, there is a $y \in B$ s.t. $g(y) = z$. Since f is surjective, there is an $x \in A$ such that $y = f(x)$. But then

$$z = g(y) = g(f(x)) = g \circ f(x)$$



$x = (g \circ f)^{-1}(z)$

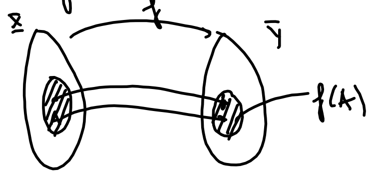
What is the inverse function? $x = f^{-1}(y) = f^{-1}(g^{-1}(z))$
$= f^{-1} \circ g^{-1}(z)$

$x = (g \circ f)^{-1}(z)$ and $x = f^{-1} \circ g^{-1}(z)$

hence $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

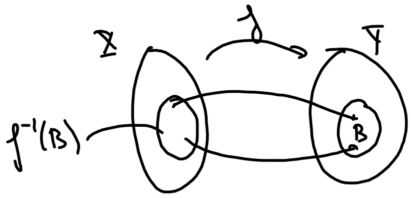
Direct and inverse images

Assume $f: X \to Y$ and that $A \subseteq X$. The image of A under f is



$$f(A) = \{f(a) : a \in A\}$$

Assume next that $B \subseteq Y$. Then the inverse image of B under f is



$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

Proposition: (i) $f^{-1}(A_1 \cup \dots \cup A_n) = f^{-1}(A_1) \cup f^{-1}(A_2) \cup \dots \cup f^{-1}(A_n)$

(more generally, $f^{-1}\left(\bigcup_{A \in \mathcal{A}} A\right) = \bigcup_{A \in \mathcal{A}} f^{-1}(A)$)

(ii) $f^{-1}(A_1 \cap \dots \cap A_n) = f^{-1}(A_1) \cap \dots \cap f^{-1}(A_n)$

(more generally, $f^{-1}\left(\bigcap_{A \in \mathcal{A}} A\right) = \bigcap_{A \in \mathcal{A}} f^{-1}(A)$)

(iii) $f^{-1}(\underbrace{A^c}_{Y \setminus A}) = \underbrace{\left(f^{-1}(A)\right)^c}_{X \setminus f^{-1}(A)}$

Proof of (i): $x \in \underbrace{f^{-1}(A_1 \cup \dots \cup A_n)} \Leftrightarrow f(x) \in A_1 \cup \dots \cup A_n$

$\Leftrightarrow f(x) \in A_i$ for at least one $i \Leftrightarrow x \in f^{-1}(A_i)$ for at least one i

$\Leftrightarrow x \in \underbrace{f^{-1}(A_1) \cup f^{-1}(A_2) \cup \dots \cup f^{-1}(A_n)}$

Proposition: (i) $f(A_1 \cup \dots \cup A_n) = f(A_1) \cup \dots \cup f(A_n)$

(more generally, $f\left(\bigcup_{A \in \mathcal{A}} A\right) = \bigcup_{A \in f} f(A)$)

(ii) $f(A_1 \cap \dots \cap A_n) \subseteq f(A_1) \cap \dots \cap f(A_n)$

(more generally, $f\left(\bigcap_{A \in \mathcal{A}} A\right) \subseteq \bigcap_{A \in \mathcal{A}} f(A)$)

Example of inequality in (ii): $X = \{x_1, x_2\}$, $Y = \{y\}$

$f: X \to Y$: $f(x_1) = f(x_2) = y$

$A_1 = \{x_1\}$, $A_2 = \{x_2\}$

$f(A_1 \cap A_2) = \emptyset$. $f(A_1 \cap A_2) \subsetneq f(A_1) \cap f(A_2)$

$\underbrace{f(A_1)}_{\{f(x_1)\}} \cap \underbrace{f(A_2)}_{\{f(x_2)\}} = \{y\} \cap \{y\} = \{y\}$

Countability

Def: A set A is countable if there is a list
$a_1, a_2, a_3, \dots, a_n, \dots$
containing all elements of A.

Example: A finite set $A = \{a_1, a_2, \dots, a_n\}$

List $a_1, a_2, \dots, a_n, a_1, a_1, a_1, \dots$

Example: $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

$a_1 = 1, a_2 = 2, a_3 = 3, \dots$

Example: $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$

Proposition. If A and B are countable then

$$A \times B = \{(a,b) : a \in A, b \in B\}$$

is countable.

Proof: Let $A = \{a_1, a_2, a_3, \dots, a_n, \dots\}$
$B = \{b_1, b_2, b_3, \dots, b_n, \dots\}$

Then

$$A \times B = \{ \underbrace{(a_1,b_1)}_{\text{sum } 2}, \underbrace{(a_2,b_1), (a_1,b_2)}_{\text{sum } 3}, \underbrace{(a_3,b_1), (a_2,b_2), (a_1,b_3)}_{\text{sum } 4},$$

Theorem: $\mathbb{Q}$ is countable

Proof: Recall $\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$

$\mathbb{Z} = \{m_1, m_2, m_3, \dots\}$

$\mathbb{N} = \{n_1, n_2, n_3, \dots\}$

Then

$$\mathbb{Q} = \left\{ \frac{m_1}{n_1}, \frac{m_2}{n_1}, \frac{m_1}{n_2}, \frac{m_3}{n_1}, \frac{m_2}{n_2}, \frac{m_1}{n_3}, \dots \right\}$$

Theorem: $\mathbb{R}$ is not countable.

Proof: (Cantor's diagonal argument). Assume for contradiction that $\mathbb{R}$ is countable. Then we have a listing $r_1, r_2, \dots$

$r_1 = \_\_\_\_.\, r_{11}\, r_{12}\, r_{13}\, r_{14} \dots\dots$

$r_2 = \_\_\_\_.\, r_{21}\, r_{22}\, r_{23}\, r_{24}$

$r_3 \quad \_\_\_\_.\, r_{31}\, r_{32}\, r_{33}\, r_{34}$

$r_4 \quad \_\_\_\_.\, r_{41}\, r_{42}\, r_{43}\, r_{44}$

$\vdots$

$r = 0.d_1 d_2 d_3 d_4 \dots$

$$d_1 = \begin{cases} 1 & \text{if } r_{11} \neq 1 \\ 2 & \text{if } r_{11} = 1 \end{cases}$$

$$d_2 = \begin{cases} 1 & \text{if } r_{22} \neq 1 \\ 2 & \text{if } r_{22} = 1 \end{cases}$$

$\vdots$

This r is not on the list since it differs from the list at some decimal. This is a contradiction since all real numbers were supposed to be on the list.