Reminder:

$f \in D \quad f : [-\pi, \pi] \to \mathbb{C}$

Fourier approximations

$$s_N = \sum_{n=-N}^{N} \alpha_n e_n \quad \text{where} \quad \alpha_n = \langle f, e_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$$

Want to prove that $s_N \to f$, but in what sense?

Aiming for pointwise convergence $s_N(x) \to f(x)$ for "most" x.

Cesaro convergence

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. We say that $\{a_n\}$ converges to a in Cesaro sense/mean if

$$\frac{a_0 + a_1 + \dots + a_{n-1}}{n} \longrightarrow a$$

Example: $\{1, 0, 1, 0, 1, 0, \dots\}$ converges $1/2$ in Cesaro sense.

Proposition: If $\{a_n\}$ converges to a, then $\{a_n\}$ also converges to a in Cesaro sense.

Idea: Perhaps we should check if $\{s_N(x)\}$ converges to f in Cesaro sense, i.e.

$$S_n(x) = \frac{s_0(x) + s_1(x) + \dots + s_{n-1}(x)}{n} \to f(x). \text{ for all } x.$$

Recall the Dirichlet kernel $D_k(x)$:

$$s_k(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u) D_k(u)\,du, \text{ where } D_k(u) = \frac{\sin[(k+\frac{1}{2})u]}{\sin\frac{u}{2}}$$

This means that

$$S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} s_k(x) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u) D_k(u)\,du$$

$$= \frac{1}{2\pi n} \int_{-\pi}^{\pi} f(x-u) \sum_{k=0}^{n-1} D_k(u)\,du$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u) \underbrace{\frac{1}{n} \sum_{k=0}^{n-1} D_k(u)}_{F_n(u) \text{ - Fejér-kernel.}}\,du$$

Let us compute the Fejér-kernel:

$$F_n(u) = \frac{1}{n} \sum_{k=0}^{n-1} D_k(u) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{\sin[(k+\frac{1}{2})u]}{\sin\frac{u}{2}} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$= \frac{1}{n \sin\frac{u}{2}} \sum_{k=0}^{n-1} \frac{e^{i(k+\frac{1}{2})u} - e^{-i(k+\frac{1}{2})u}}{2i}$$

$$= \frac{1}{2in\sin\frac{u}{2}} \left(\sum_{k=0}^{n-1} e^{i(k+\frac{1}{2})u} - \sum_{k=0}^{n-1} e^{-i(k+\frac{1}{2})u} \right)$$

Let's look at the sums separately:

$$\sum_{k=0}^{n-1} e^{i(k+\frac{1}{2})u} = e^{i\frac{u}{2}} \sum_{k=0}^{n-1} (e^{iu})^k \overset{\text{Geometric series}}{=} e^{i\frac{u}{2}} \frac{1 - e^{inu}}{1 - e^{iu}}$$

$$= \frac{1 - e^{inu}}{e^{-i\frac{u}{2}}(1 - e^{iu})} = \frac{1 - e^{inu}}{e^{-i\frac{u}{2}} - e^{i\frac{u}{2}}} = \frac{e^{inu} - 1}{e^{i\frac{u}{2}} - e^{-i\frac{u}{2}}}.$$

and

$$\sum_{k=0}^{n-1} e^{-i(k+\frac{1}{2})u} = e^{-i\frac{u}{2}} \sum_{k=0}^{n-1} (e^{-iu})^k \overset{\text{Geometric series}}{=} e^{-i\frac{u}{2}} \frac{1 - e^{-inu}}{1 - e^{-iu}}$$

$$= \frac{1 - e^{-inu}}{e^{i\frac{u}{2}}(1 - e^{-iu})} = \frac{1 - e^{-inu}}{e^{i\frac{u}{2}} - e^{-i\frac{u}{2}}}$$

Hence

$$F_n(u) = \frac{1}{2in\sin\frac{u}{2}} \left(\frac{e^{inu} - 1}{e^{i\frac{u}{2}} - e^{-i\frac{u}{2}}} - \frac{1 - e^{-inu}}{e^{i\frac{u}{2}} - e^{-i\frac{u}{2}}} \right)$$

$$= \frac{1}{2in\sin\frac{u}{2}} \frac{e^{inu} - 2 + e^{-inu}}{e^{i\frac{u}{2}} - e^{-i\frac{u}{2}}}$$

$$= \frac{1}{2in\sin\frac{u}{2}} \frac{(e^{i\frac{nu}{2}} - e^{-i\frac{nu}{2}})^2}{e^{i\frac{u}{2}} - e^{-i\frac{u}{2}}}$$

$$= \frac{1}{n\sin\frac{u}{2}} \frac{\left(\frac{e^{i\frac{nu}{2}} - e^{-i\frac{nu}{2}}}{2i}\right)^2}{\frac{e^{i\frac{u}{2}} - e^{-i\frac{u}{2}}}{2i}} = \frac{1}{n\sin\frac{u}{2}} \frac{\sin^2\frac{nu}{2}}{\sin\frac{u}{2}}$$

$$= \frac{\sin^2\frac{nu}{2}}{n\sin^2\frac{u}{2}}$$

Proposition: The Fejér kernel is given by

$$F_n(u) = \frac{1}{n} \sum_{k=0}^{n-1} D_k(u) = \frac{\sin^2\frac{nu}{2}}{n\sin^2\frac{u}{2}}.$$

and has the following properties:

(i) $F_n(u)$ is an even function

(ii) $0 \le F_n(u) \le$

(iii) $\frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(u)\,du = 1$

Proof: (i) $F_n(-u) = F_n(u)$ obvious.

(ii) Since $F_n(u) = \frac{\sin^2(\frac{nu}{2})}{n\sin^2\frac{u}{2}}$, we have

$$0 \le F_n(u) = \frac{1}{n} \sum_{k=0}^{n-1} D_k(u) \le \frac{1}{n} \sum_{k=0}^{n-1} D_k(0) = \frac{1}{n} \sum_{k=0}^{n-1} (2k+1)$$

$$= \frac{1}{n} \left(2 \sum_{k=0}^{n-1} k + n \right) = \frac{1}{n} \left(\cancel{2} \frac{(n-1)n}{\cancel{2}} + n \right) = \frac{n^2}{n} = n.$$

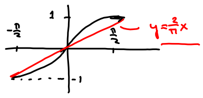
(iii) $$\frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(u)\,du = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{n} \sum_{k=0}^{n-1} D_k(u)\,du$$

$$= \frac{1}{n} \sum^{n-1} \frac{1}{2\pi} \int^{\pi} D_k(u)\,du = \frac{1}{n} \cdot n = 1$$

Lemma: For $u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $u \neq 0$, we have

$$F_n(u) \leq \frac{\pi^2}{n u^2}.$$

Proof: Note that (drawing)



$$\underbrace{|\sin x| \geq \frac{2}{\pi}|x|}$$

Hence

$$F_n(u) = \frac{\sin^2\left(\frac{nu}{2}\right)}{n \underbrace{\sin^2 \frac{u}{2}}_{\substack{\vee\!\shortmid \\ \left(\frac{2}{\pi}\frac{u}{2}\right)^2}}} \underset{\uparrow}{\leq} \frac{1}{n\left(\frac{2}{\pi}\frac{u}{2}\right)^2} = \frac{\pi^2}{n u^2}$$

Fejer's Theorem: For any $f \in D$ and all $x \in [-\pi, \pi]$, we have

$$S_n(x) \to f(x)$$

If f is continuous on $[a,b] \subseteq [-\pi,\pi]$, then the convergence is uniform on $[a,b]$

Preliminaries:

$$S_n(x) = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u) F_n(u)\, du}$$

If we put $t = -u$, then $dt = -du$ and

$$S_n(x) = \frac{1}{2\pi}\int_{\pi}^{-\pi} f(x+t) \overbrace{F_n(-t)}^{F_n(t)} (-dt)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t) F_n(t)\, dt$$

$$= \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+u) F_n(u)\, du}$$

Hence

$$S_n(x) = \frac{1}{4\pi}\int_{-\pi}^{\pi} (f(x+u) + f(x-u)) F_n(u)\, du$$

Also

$$f(x) = f(x) \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(u)\, du}_{1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) F_n(u)\, du$$

Hence

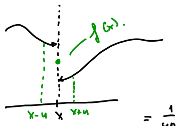
$$|S_n(x) - f(x)| = \left| \frac{1}{4\pi}\int_{-\pi}^{\pi} [f(x+u) + f(x-u) - 2f(x)] F_n(u)\, du \right|$$

To prove Fejer's theorem, we need to show that given an $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that if $n \geq N$, then

$$|S_n(x) - f(x)| < \varepsilon.$$

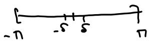
Since $f \in D$, there is $\delta > 0$ such that if $|u| < \delta$, then

$$|f(x+u) + f(x-u) - 2f(x)| < \varepsilon. \quad \text{Hence}$$



$$|S_N(x) - f(x)|$$

$$\leq \frac{1}{4\pi}\int_{-\pi}^{\pi} |f(x+u) + f(x-u) - 2f(x)| F_n(u)\, du$$



$$= \frac{1}{4\pi}\Big(\int_{-\pi}^{-\delta} |f(x+u) + f(x-u) - 2f(x)| F_n(u)\, du$$

$$+ \int_{-\delta}^{\delta} \overbrace{|f(x+u) + f(x-u) - 2f(x)|}^{<\varepsilon} F_n(u)\, du$$

$$+ \int_{\delta}^{\pi} |f(x+u) + f(x-u) - 2f(x)| F_n(u) \Big)$$

$$\leq \frac{1}{4\pi}\int_{-\delta}^{\delta} \varepsilon F_n(u)\, du + \frac{1}{4\pi}\int_{-\pi}^{-\delta} \cdots\cdots + \frac{1}{4\pi}\int_{\delta}^{\pi} \cdots\cdots$$

$$\leq \frac{\varepsilon}{2} + \underbrace{\frac{1}{4\pi}\int_{-\pi}^{-\delta} \cdots}_{\leq \frac{\varepsilon}{4}} + \underbrace{\frac{1}{4\pi}\int_{\delta}^{\pi} \cdots}_{\leq \frac{\varepsilon}{4}} < \varepsilon.$$

Let us take a look at the last integral

$$\frac{1}{4\pi}\int_{\delta}^{\pi} \overbrace{[f(x+u) + f(x-u) - 2f(x)]}^{4\|f\|_\infty} F_n(u)\, du$$

$$\leq \frac{\|f\|_\infty}{\pi}\int_{\delta}^{\pi} F_n(u)\, du \leq \frac{\|f\|_\infty}{\pi}\int_{\delta}^{\pi} \frac{\pi^2}{n u^2}\, du$$

$$\leq \frac{\|f\|_\infty}{\pi}\int_{\delta}^{\pi} \frac{\pi^2}{n \delta^2}\, du = \|f\|_\infty \frac{\pi^2}{n\delta^2} < \frac{\varepsilon}{4} \text{ by choosing } n \text{ big enough.}$$

This proves pointwise convergence.

What uniform convergence on $[a,b]$

If I can use the same δ for all x's in $[a,b]$, I get uniform convergence. But this is guaranteed by uniform continuity.

Corollary: If $f \in C_p$, then S_n converges uniformly to f, and since the S_n are trigonometric polynomials, this means that the trigonometric polynomials are dense in C_p.