Weierstrass M-test: Assume that $\sum_{n=0}^{\infty} v_n(x)$ is a series of functions.
Assume that there is a convergent series $\sum_{n=0}^{\infty} M_n$ of numbers such that for all $x \in A$, we have

$$|v_n(x)| \leq M_n.$$

Then $\sum_{n=0}^{\infty} v_n(x)$ converges uniformly on A.

Proof: Pick $x \in A$. Let $s_N(x) = \sum_{n=0}^{N} v_n(x)$ and $S_N = \sum_{n=0}^{N} M_n$

Note that if $M > N$, then

$$|s_M(x) - s_N(x)| = \left| \sum_{n=0}^{M} v_n(x) - \sum_{n=0}^{N} v_n(x) \right| = \left| \sum_{n=N+1}^{M} v_n(x) \right|$$

$$\leq \sum_{n=N+1}^{M} |v_n(x)| \leq \sum_{n=N+1}^{M} M_n = S_M - S_N$$ ← can get this as small as I want since $\{S_N\}$ converges and hence is a Cauchy sequence

Hence $\{s_N(x)\}$ is a Cauchy sequence and hence converges to limit $s(x)$.

This shows $\{s_N(x)\}$ converges pointwise to $s(x) = \sum_{n=0}^{\infty} v_n(x)$ on A.

But this is convergence uniform? For any $x \in A$:

$$|s(x) - s_N(x)| = \left|\sum_{n=N+1}^{\infty} v_n(x)\right| \leq \sum_{n=N+1}^{\infty} M_n = \sum_{n=0}^{\infty} M_n - \sum_{n=0}^{N} M_n$$

$$= S - S_N$$

$\leq \varepsilon$ by choosing N large enough.

Hence $s_N(x)$ converges to $s(x)$ on A.

Example: Show that for any $a > 0$, the series

$$\sum_{n=1}^{\infty} \underbrace{n e^{-nx}}_{v_n(x)}$$

converges uniformly on $[a, \infty)$

We are going to use the Weierstrass M-test, so we need a convergent series $\sum_{n=1}^{\infty} M_n$ such that $M_n \geq v_n(x)$ for all $x \in [a, \infty)$

Note that $v_n(x) = ne^{-nx}$ is decreasing on $[a, \infty)$, and thus $|v_n(x)| \leq \underbrace{ne^{-na}}_{M_n}$ for all $x \in [a, \infty)$. We choose $M_n = ne^{-na}$, and since $|v_n(x)| \leq M_n$, we only need to prove that $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} ne^{-na}$ converges.

Ratio test: $\lim_{n\to\infty} \frac{M_{n+1}}{M_n} = \lim_{n\to\infty} \frac{(n+1)e^{-(n+1)a}}{ne^{-na}} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \frac{\cancel{e^{-na}} \cdot e^{-a}}{\cancel{e^{-na}}}$

$= 1 \cdot e^{-a} < 1$

Hence the series $\sum M_n$ converges, and by Weierstrass' M-test, the sequence $\sum_{n=1}^{\infty} ne^{-nx}$ converges uniformly on $[a, \infty)$. No

Does this mean that $\sum_{n=1}^{\infty} ne^{-nx}$ converges uniformly on $(0, \infty)$?

NO!

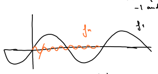
Differentiation

Question: If $f_n \to f$ uniformly, will $f_n' \to f'$?

No

Examples: $f_n(x) = \frac{\sin(nx)}{n}$ converges uniformly $f(x) = 0$

but $f_n'(x) = \frac{\cos(nx) \cdot \cancel{n}}{\cancel{n}} = \cos nx$ oscillating between -1 and 1 $\quad f'(x) = 0$



Theorem: Assume that $\{f_n\}$ is a sequence of functions whose derivatives are continuous on $[a,b]$. Assume that $\{f_n'\}$ converges uniformly on $[a,b]$ to g. Assume further that $\{f_n(x_0)\}$ converges for at least one $x_0 \in [a,b]$. Then the original sequence $\{f_n\}$ converges uniformly to a function f such that $f' = g$.

$\{f_n\} \longrightarrow f$ (uniformly)
$\Uparrow \quad f' = g$
$\{f_n'\} \longrightarrow g$ (uniform)

Proof: Since $f_n' \to g$ uniformly on $[a,b]$, we have by the corollary at the start of the lecture that

$$\int_{x_0}^{x} f_n'(t)\,dt \longrightarrow \int_{x_0}^{x} g(t)\,dt \quad \text{(uniformly)}$$

Fund Th Calculus

$$f_n(x) - \underbrace{f_n(x_0)}_{\downarrow\, d} \longrightarrow \int_{x_0}^{x} g(t)\,dt$$

$$f_n(x) \to d + \int_{x_0}^{x} g(t)\,dt \quad \text{(uniform)}$$

Hence $f_n(x)$ converges uniformly to $\underbrace{d + \int_{x_0}^{x} g(t)\,dt}_{f(x)}$

Hence

$$f'(x) = \left(d + \int_{x_0}^{x} g(t)\,dt\right)' = 0 + g(x) = g(x)$$

Theorem: Assume that $\sum_{n=0}^{\infty} v_n(x)$ is a series of functions whose derivatives $v_n'(x)$ are continuous on $[a,b]$. Assume that the "differentiated series" $\sum_{n=0}^{\infty} v_n'(x)$ converges uniformly on $[a,b]$ and $\sum_{n=0}^{\infty} v_n(x_0)$ converges for at least one $x_0 \in [a,b]$. Then $\sum_{n=0}^{\infty} v_n(x)$ converges uniformly on $[a,b]$, and

$$\left(\sum_{n=0}^{\infty} v_n(x)\right)' = \sum_{n=0}^{\infty} v_n'(x).$$

Proof: Put $s_N(x) = \sum_{n=0}^{N} v_n(x)$, then $s_N'(x) = \sum_{n=0}^{N} v_n'(x)$.

Now $\sum_{n=0}^{\infty} v_n'(x)$ converging uniformly to $g(x)$, means that $s_N'(x) \to g(x)$ uniformly. Hence by the theorem for sequences

$s_N(x) \to f(x)$ uniformly, and $f'(x) = g(x)$.

This means that $f(x) = \lim s_N(x) = \sum_{n=0}^{\infty} v_n(x)$ exists, and that

$$f'(x) = \left(\sum_{n=0}^{\infty} v_n(x)\right)' = g(x) = \lim_{N\to\infty} s_N'(x) = \sum_{n=0}^{\infty} v_n'(x)$$

Powers series: $v_n(x) = c_n (x-a)^n$