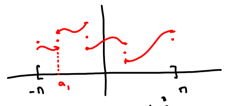
Review Ch. 10 and Ch 6

D consists of all functions $f: [-\pi,\pi] \to \mathbb{C}$ such that there are points $-\pi = a_0 < a_1 < \dots < a_n = \pi$ such that

(i) f is continuous on the intervals (a_{i-1}, a_i)

(ii) The one-sided limits $\lim_{x \to a_i^-} f(x) = f(a_i^-)$ and $\lim_{x \to a_i^+} f(x) = f(a_i^+)$ exist.

(iii) $f(a_i) = \dfrac{f(a_i^+) + f(a_i^-)}{2}$



Inner product: $\langle f, g \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$

Corresponding norm: $\|f\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx\right)^{1/2}$

Orthonormal basis: $\{e_n\}_{n\in\mathbb{Z}}$ $\quad e_n(x) = e^{inx}$

Fourier coefficients: $\alpha_n = \langle f, e_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$

Fourier series: $\sum_{n=-\infty}^{\infty} \alpha_n e^{inx}$

When is $f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{inx}$ and in what sense.

Partial sums: $s_n(x) = \sum_{k=-n}^{n} \alpha_k e^{ikx}$

(i) For all f in D $f = \sum_{n=-\infty}^{\infty} \alpha_n e^{inx}$ in L^2-sense:

$\|f - s_n\|_2 \to 0$ as $n \to \infty$. (i.e.)

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - s_n(x)|^2\,dx\right)^{1/2} \to 0$$

(ii) **Cesaro-convergence:** $S_n(x) = \dfrac{s_0(x) + s_1(x) + \dots + s_{n-1}(x)}{n}$

Then for all $f \in D$, $S_n(x) \to f(x)$ for all x.
If f is continuous $[a,b] \subseteq [-\pi,\pi]$, then the convergence is uniform on $[a,b]$.

(iii) **Dini's Test:** Assume $f \in D$ and that for an $x \in [-\pi,\pi]$ we have that

$$\int_{-\delta}^{\delta} \left| \frac{f(x+u) + f(x-u) - 2f(x)}{u} \right| du < \infty$$

for some $\delta > 0$. Then $s_n(x) \to f(x)$.

Corollary: If f is differentiable at x, then $s_n(x) \to f(x)$

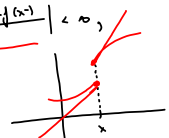
Corollary: Assume

$$\lim_{u \to 0^+} \left| \frac{f(x+u) - f(x^+)}{u} \right| < \infty$$

and

$$\lim_{u \to 0^-} \left| \frac{f(x+u) - f(x^-)}{u} \right| < \infty,$$

then $s_n(x) \to f(x)$



Dirichlet kernel: $D_N(u) = \sum_{n=-N}^{N} e^{inu} = \dfrac{\sin[(N+\frac{1}{2})u]}{\sin[\frac{u}{2}]}$

$$s_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)D_N(u)\,du$$

Fejer kernel: $F_N(u) = \dfrac{D_0(u) + \dots + D_{N-1}(u)}{N} = \dfrac{\sin^2[\frac{Nu}{2}]}{N\sin^2\frac{u}{2}}$

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)F_N(u)\,du$$

Real Fourier Series

$e^{inx} = (\cos nx + i\sin nx)$

$f: [-\pi,\pi] \to \mathbb{R}$

Real Fourier series: $\dfrac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

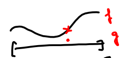
$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

The same theorems of convergence apply in this case!

Exam 2021, no 6: Assume $f, g \in D$ and that

$$\int_{-\pi}^{\pi} f(x)e^{inx}\,dx = \int_{-\pi}^{\pi} g(x)e^{inx}\,dx \quad \text{for all } n \in \mathbb{Z}$$

Prove that $f = g$.



Need to prove that $\underbrace{f - g}_{h} = 0$, and we know that

$$0 = \int_{-\pi}^{\pi} f(x)e^{inx}\,dx - \int_{-\pi}^{\pi} g(x)e^{inx}\,dx = \int_{-\pi}^{\pi} (\underbrace{f(x) - g(x)}_{h})e^{inx}\,dx$$

$$= \int_{-\pi}^{\pi} h(x)e^{inx}\,dx$$

It suffices to prove that if $h \in D$ and $\int_{-\pi}^{\pi} h(x)e^{inx}\,dx = 0$ for all n, then $h = 0$

Note

$$\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)e^{-inx}\,dx = \frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi} h(x)e^{i(-n)x}\,dx} = 0.$$

Hence $h \in D$ has all it's Fourier

Chapter 6: Derivatives

Derivative of a function $\vec{F}$ at a point $\vec{a}$. This is **the** linear map A such that

$$\vec{\sigma}(\vec{r}) = \vec{F}(\vec{a}+\vec{r}) - \vec{F}(\vec{a}) - A(\vec{r})$$

goes to 0 faster than $\vec{r}$, which means $\frac{\|\sigma(\vec{r})\|}{\|\vec{r}\|} \to 0$ as $\vec{r} \to 0$.

If such an A exists, we say that $\vec{F}$ is differentiable at $\vec{a}$ and that $A = \vec{F}'(\vec{a})$ is the derivative.

Directional derivative: $\vec{F}'(\vec{a};\vec{r}) = \lim_{t\to 0} \frac{\vec{F}(\vec{a}+t\vec{r}) - \vec{F}(\vec{a})}{t}$ (Gateaux derivative)

Th: If $\vec{F}$ is differentiable at $\vec{a}$, then $\vec{F}'(\vec{a})(\vec{r}) = \vec{F}'(\vec{a};\vec{r})$

Exam 2018, Problem 1: $L: C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$

function

$L(f)$

function

$$L(f)(x) = \int_0^x f(s)^3\,ds$$

a) Find the directional derivative $L'(f;r)$:

$$L'(f;r)(x) = \lim_{t\to 0} \frac{L(f+tr)(x) - L(f)(x)}{t}$$

$$= \lim_{t\to 0} \frac{\int_0^x (f(s)+tr(s))^3\,ds - \int_0^x f(s)^3\,ds}{t}$$

$$= \lim_{t\to 0} \frac{\int_0^x [\cancel{f(s)^3} + 3f(s)^2 \cancel{t} r(s) + 3f(s)t^{\cancel{2}}r(s)^2 + t^{\cancel{3}\,2}r(s)^3]\,ds - \int_0^x \cancel{f(s)^3}\,ds}{\cancel{t}}$$

$$= \lim_{t\to 0} \int_0^x [3f(s)^2 r(s) + 3f(s)\,t\,r(s)^2 + t^2 r(s)^3]\,ds = \int_0^x 3f(s)^2 r(s)\,ds$$

b) Prove that L is differentiable and find $L'(f)(r)$

Candidate for derivative $A(r)(x) = \int_0^x 3f(s)^2 r(s)\,ds$

Check:

$$\sigma(r)(x) = L(f+r)(x) - L(f)(x) - A(r)(x)$$

$$= \int_0^x (f(s)+r(s))^3\,ds - \int_0^x f(s)^3\,ds - \int_0^x 3f(s)^2 r(s)\,ds$$

$$= \int_0^x (\cancel{f(s)^3} + \cancel{3f(s)^2 r(s)} + 3f(s)r(s)^2 + r(s)^3)\,ds - \int_0^x \cancel{f(s)^3}\,ds - \int_0^x \cancel{3f(s)^2 r(s)}\,ds$$

$$= \int_0^x (3f(s)r(s)^2 + r(s)^3)\,ds \le \int_0^x (3\|f\|\|r\|^2 + \|r\|^3) \le (3\|f\| + \|r\|)\|r\|^2$$

Hence for all x: $\sigma(r)(x) \le (3\|f\| + \|r\|)\|r\|^2$

$$\|\sigma(r)\| \le (3\|f\| + \|r\|)\|r\|^2$$

$$\frac{\|\sigma(r)\|}{\|r\|} \le (3\|f\| + \|r\|)\|r\| \to 0$$

Left you

Hence $\sigma(r)$ goes to 0 faster than r.

This means that if my candidate $A(r)(x) = \int_0^x 3f(s)^2 r(s)\,ds$ is a bounded linear, then $\vec{F}$ is differentiable and $\vec{F}'(\vec{a})(\vec{r})(x) = \int_0^x 3f(s)^2 r(s)\,ds$ is the derivative.

Chain Rule: $(\vec{F}\circ\vec{G})(\vec{a}) = \vec{F}'(\vec{G}(a)) \circ \vec{G}'(a)$

Taylor Series:

Inverse Function Theorem

Exam: Carbon paper