Power Series

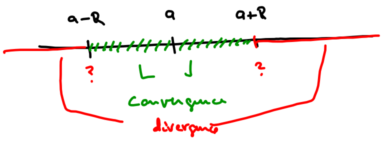
$$\sum_{n=0}^{\infty} c_n (x-a)^n, \quad a, c_n \in \mathbb{R}$$

Review: Three cases

(i) Series converges for all x $\quad R = \infty$

(ii) Series only converges for $x = a$ $\quad R = 0$

(iii) There is $R > 0$ such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$. (R radius of convergence)



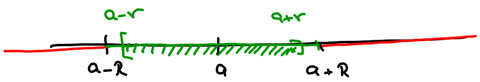
It turns out that:

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{1}{R} \quad \text{with the convention that} \begin{cases} \frac{1}{0} = \infty \\ \frac{1}{\infty} = 0 \end{cases}$$

Theorem: Let

$$\limsup \sqrt[n]{|c_n|} = \frac{1}{R}$$

If $|x-a| > R$, then the series $\sum c_n (x-a)^n$ diverges. If $|x-a| < R$, then the series $\sum c_n (x-a)^n$ converges, and the convergence is uniform on each interval $[a-r, a+r]$ for all $0 \le r < R$



Proof: First consider the case where $|x-a| > R$; i.e. $\frac{1}{|x-a|} < \frac{1}{R}$

Since $\limsup \underbrace{\sqrt[n]{|c_n|}} = \frac{1}{R} > \underbrace{\frac{1}{|x-a|}}$, there must be arbitrary larger n such that $\sqrt[n]{|c_n|} > \frac{1}{|x-a|}$. Hence

$\sqrt[n]{|c_n|}\, |x-a| > 1 \Rightarrow |c_n (x-a)^n| > 1$. Hence the terms of $\sum c_n (x-a)^n$ do not go to 0, and hence the series diverges.

For the converse, let $0 < r < R$. We want to prove uniform convergence $[a-r, a+r]$. Since $r < R$, we have $\underline{\frac{1}{r} > \frac{1}{R}}$.

Choose a $\underline{b < 1}$ such that $\frac{b}{r} > \frac{1}{R}$. Now since

$$\limsup \underbrace{\sqrt[n]{|c_n|}} = \frac{1}{R} < \underline{\underline{\frac{b}{r}}},$$

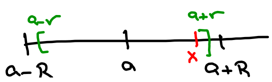
there must be a N such that when $\underline{n \ge N}$, $\sqrt[n]{|c_n|} < \frac{b}{r}$.

Hence $|c_n| r^n < b^n$ for $n \ge N$. If $x \in [a-r, a+r]$, then

$\underline{\underline{\sum_{n=N}^{\infty} c_n (x-a)^n}}$ then $|c_n (x-a)^n| < b^n$, where $\sum_{n=N}^{\infty} b^n$ is a convergent geometric series (as $b < 1$)

By Weierstrass M-test, $\sum_{n=N}^{\infty} c_n (x-a)^n$ converges uniformly on $[a-r, a+r]$. This means that $\sum_{n=0}^{\infty} a_n (x-a)^n$ converges uniformly on $[a-r, a+r]$ for any $r < R$.

Since any $x \in (a-R, a+R)$ is in an interval $[a-r, a+r]$, the series converges for all $x \in (a-R, a+R)$



Observation:

$$\sum_{n=0}^{\infty} c_n (x-a)^n \qquad \frac{1}{R} = \limsup \sqrt[n]{|c_n|}$$

$$\sum_{n=0}^{\infty} c_n (x-a)^{n+1} \quad \text{same radius of convergence } R$$

$$\|$$

$$\sum_{n=1}^{\infty} c_{n-1}(x-a)^n , \qquad \frac{1}{R} = \limsup \sqrt[n]{|c_{n-1}|}$$

$$\sum_{n=1}^{\infty} c_n (x-a)^{n-1} \qquad R \text{ same as before}$$

$$\|$$

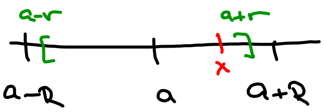
$$\sum_{n=0}^{\infty} c_{n+1} (x-a)^n \qquad \frac{1}{R} = \limsup \sqrt[n]{|c_{n+1}|}$$

Theorem: Assume that $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ ~~so~~ has a radius of convergence $R>0$. Then

$$\int_a^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} \qquad \text{for } x \in (a-R, a+R)$$

where the integrated series has radius of convergence R (but we may have convergence at endpoints.

Proof. Let $x \in (a-R, a+R)$, and choose r such that $x \in [a-r, a+r]$. Since the power series converges uniformly on $[a-r, a+r]$, we can integrate term by term on this interval



$$\int_a^x f(t)\,dt = \sum_{n=0}^{\infty} \int_a^x c_n (t-a)^n\,dt = \sum_{n=0}^{\infty} \left[c_n \frac{(t-a)^{n+1}}{n+1} \right]_a^x$$

$$= \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}.$$

Example: $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x|<1$.

Then

$$\int_0^x \frac{1}{1-t}\,dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$\|$$

$$\underbrace{-\ln|1-x| = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}}_{\text{does this hold for } x=-1\text{?}} \qquad \text{interval of convergence } [-1,1)$$

gained convergence at $x=-1$

Theorem: Assume that $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius R of convergence larger than 0. Then

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

for all $x \in (x-R, x+R)$. The differentiated series $\sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$ has the same radius of convergence R, but we may lose convergence at end points.

Proof: The differentiated series

$$\sum_{n=1}^{\infty} n c_n (x-a)^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} (x-a)^n$$

has radius of convergence

$$\frac{1}{R'} = \limsup_{n\to\infty} \sqrt[n]{(n+1)|c_{n+1}|} = \limsup_{n\to\infty} \underbrace{\sqrt[n]{n+1}}_{\downarrow \text{L'H} \atop 1} \underbrace{\sqrt[n]{|c_{n+1}|}}_{\downarrow \atop \frac{1}{R} \text{ of the original series.}}$$

which is the same as for the original series. This means that for any $r<R$, the differentiated series converges uniformly on $[a-r, a+r]$.

Hence by last time

$$(*) \qquad f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \quad \text{for all } x \in [a-r, a+r]$$

Since any x in $(a-R, a+R)$ is in an interval $[a-r, a+r]$, we $(*)$ for all x in $(a-R, a+R)$.

Abel's summation formula: Assume that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of real numbers and let $s_n = \sum_{i=0}^{n} a_i$. Then

$$\sum_{n=0}^{N} a_n b_n = s_N b_N + \sum_{n=0}^{N-1} s_n (b_n - b_{n+1})$$

If $\{s_N\}$ converges and $b_N \to 0$, then

$$\sum_{n=0}^{\infty} a_n b_n = \sum_{n=0}^{\infty} s_n (b_n - b_{n+1})$$

Summation by parts:

$$\int u v' \, dx = uv - \int u' v \, dx$$

Proof: Note that

$a_n = s_n - s_{n-1}$ for ~~n≥1~~ for $n=0$ if we put $s_{-1} = 0$.

And then

$$\sum_{n=0}^{N} a_n b_n = \sum_{n=0}^{N} (s_n - s_{n-1}) b_n = \sum_{n=0}^{N} s_n b_n - \sum_{n=1}^{N} s_{n-1} b_n$$

$$= \sum_{n=0}^{N} s_n b_n - \sum_{n=0}^{N-1} s_n b_{n+1} = s_N b_N + \sum_{n=0}^{N-1} s_n b_n - \sum_{n=0}^{N-1} s_n b_{n+1}$$

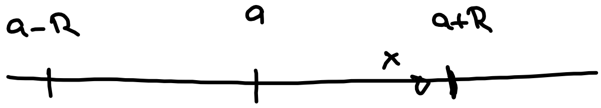
$$= s_N b_N + \sum_{n=0}^{N-1} s_n (b_n - b_{n+1})$$

Assuming s_N converges and $b_N \to 0$, we can take the limits.

$$\sum_{n=0}^{\infty} a_n b_n = \sum_{n=0}^{\infty} s_n (b_n - b_{n+1})$$

Abel's Theorem: Assume that $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence R and that the series converges for $x = a+R$. Then f is continuous at $a+R$ in the sense that

$$\lim_{x\to a+R} f(x) = f(a+R)$$



Proof for $a=0$: Must prove that

$$\lim_{x\to R^-} f(x) = f(R); \text{ i.e. } \sum_{n=0}^{\infty} c_n x^n \to \sum_{n=0}^{\infty} c_n R^n$$

$s_n = a_0+a_1+\dots+a_n$
$= \sum_{i=0}^{n} c_i R^i$

We have

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \underbrace{c_n R^n}_{a_n} \underbrace{\left(\frac{x}{R}\right)^n}_{b_n} = \sum_{n=0}^{\infty} s_n (b_n - b_{n+1})$$

$$= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} c_i R^i\right)\left(\left(\frac{x}{R}\right)^n - \left(\frac{x}{R}\right)^{n+1}\right) = \sum_{n=0}^{\infty} \sum_{i=1}^{n} c_i R^i \left(\frac{x}{R}\right)^n \left(1-\frac{x}{R}\right)$$

$$= \left(1-\frac{x}{R}\right) \sum_{n=0}^{\infty} \left(\sum_{i=1}^{n} c_i R^i\right)\left(\frac{x}{R}\right)^n.$$

Recall that

$$\frac{1}{1-\frac{x}{R}} = \sum_{n=1}^{\infty} \left(\frac{x}{R}\right)^n \quad \Big|\ f(R)\left(1-\frac{x}{R}\right)$$

$$f(R) = \left(1-\frac{x}{R}\right) \sum_{n=0}^{\infty} f(R) \left(\frac{x}{R}\right)^n$$

Thus

$$|f(x)-f(R)| = \left(1-\frac{x}{R}\right)\left[\left[\sum_{i=1}^{n} c_i R^i - f(R)\sum_{n=0}^{\infty}\right.\right.$$