Abel's Theorem

Theorem: Assume that $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series with radius of convergence $R>0$. If the series converges at an endpoint of the interval of convergence it is continuous there. More precisely, if the series converges at $a+R$, then

$$\lim_{x\uparrow a+R} f(x) = f(a+R)$$

and if the series converges at $a-R$, then

$$\lim_{x\downarrow a-R} f(x) = f(a-R).$$

Tool: Abel summation: $\sum_{n=0}^{\infty} a_n b_n = \sum_{n=0}^{\infty} s_n (b_n - b_{n+1})$
where $s_n = \sum_{i=0}^{n} a_i$.

Proof: Assume $a=0$, and assume that the series converges at R, $\sum_{n=0}^{\infty} c_n R^n$ exists $= f(R)$.

Let $f_n(R) = \sum_{i=0}^{n} \underbrace{c_i R^i}_{a_i}$. Note

$f_n(R) \to f(R)$

We want to prove that $f(x)\to f(R)$ when $x\uparrow R$. Note that

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n R^n \frac{x^n}{R^n} = \sum_{n=0}^{\infty} \underbrace{c_n R^n}_{a_n} \underbrace{\left(\frac{x}{R}\right)^n}_{b_n}$$

$$\overset{\text{Abel summation}}{=} \sum_{n=0}^{\infty} f_n(R)\left(\left(\frac{x}{R}\right)^n - \left(\frac{x}{R}\right)^{n+1}\right)$$

$$= \sum_{n=0}^{\infty} f_n(R)\left(\frac{x}{R}\right)^n \left(1-\left(\frac{x}{R}\right)\right)$$

$$= \left(1-\left(\frac{x}{R}\right)\right)\sum_{n=0}^{\infty} f_n(R)\left(\frac{x}{R}\right)^n$$

Claim

$$f(R) = \left(1-\frac{x}{R}\right)\sum_{n=0}^{\infty} f(R)\left(\frac{x}{R}\right)^n$$

Proof of claim: $\left(1-\frac{x}{R}\right)\sum_{n=0}^{\infty} f(R)\left(\frac{x}{R}\right)^n$

$$= f(R)\left(1-\frac{x}{R}\right)\underbrace{\sum_{n=0}^{\infty}\left(\frac{x}{R}\right)^n}_{\frac{1}{1-\frac{x}{R}}} = f(R)\cancel{\left(1-\frac{x}{R}\right)}\frac{1}{\cancel{1-\frac{x}{R}}} = f(R)$$

This means

$$|f(x)-f(R)| = \left|\left(1-\frac{x}{R}\right)\sum_{n=0}^{\infty} f_n(R)\left(\frac{x}{R}\right)^n - \left(1-\frac{x}{R}\right)\sum_{n=0}^{\infty} f(R)\left(\frac{x}{R}\right)^n\right|$$

$$\le \underbrace{\left(1-\frac{x}{R}\right)}_{\downarrow\, 0}\sum_{n=0}^{\infty} |f_n(R)-f(R)|\left(\frac{x}{R}\right)^n$$

Given an $\varepsilon>0$, there is an N such that for $n\ge N$, $|f_n(R)-f(R)|<\frac{\varepsilon}{2}$. Thus

$$|f(x)-f(R)| \le \left(1-\frac{x}{R}\right)\left(\sum_{n=0}^{N} |f_n(R)-f(R)|\left(\frac{x}{R}\right)^n + \sum_{n=N+1}^{\infty} |f_n(R)-f(R)|\left(\frac{x}{R}\right)^n\right)$$

$$\le \left(1-\frac{x}{R}\right)\left(\sum_{n=0}^{N} |f_n(R)-f(R)|\left(\frac{x}{R}\right)^n + \underbrace{\sum_{n=0}^{\infty} \frac{\varepsilon}{2}\left(\frac{x}{R}\right)^n}_{\frac{\frac{\varepsilon}{2}}{1-\frac{x}{R}}}\right)$$

$$= \underbrace{\underbrace{\left(1-\frac{x}{R}\right)}_{\downarrow\, 0}\underbrace{\sum_{n=0}^{N} |f_n(R)-f(R)|\left(\frac{x}{R}\right)^n}_{\text{bounded}}} + \frac{\varepsilon}{2}$$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

by choosing x sufficiently close to