Review of Ch 4 and 5

Modes of continuous:

(Pointwise) continuity: f is continuous at all $a \in X$.

Uniform continuity: For every $\varepsilon > 0$, there is a $\delta > 0$ such that $x, y \in A$ we have $d_Y(f(x), f(y)) < \varepsilon$ whenever $d_X(x,y) < \delta$ (the same δ works for all pairs x, y).

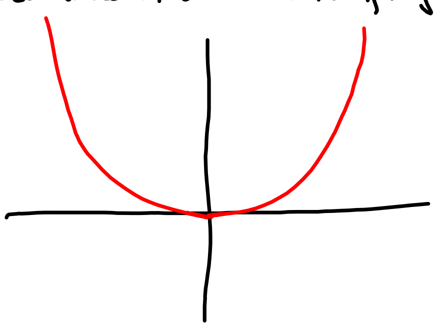
Equicontinuity: A family $\mathcal{F}$ of functions $f: X \to Y$ is equicontinuous if for every $\varepsilon > 0$ there is a $\delta > 0$ such that when $d_X(x,y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$ for $f \in \mathcal{F}$.

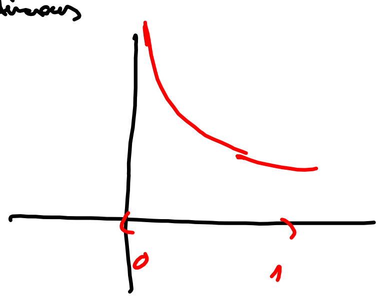
Theorem: If X is compact, all continuous $f: X \to Y$ are uniformly continuous

Arzela-Ascoli Theorem: If X is compact, a subset $\mathcal{F}$ of $C(X, \mathbb{R}^n)$ is compact if and only if it is closed, bounded, and equicontinuous.

Example: Continuous, but not uniformly continuous

$f(x) = x^2$





Modes of convergence

Pointwise convergence on A: $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in A$.

Uniform convergence on A: For every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that when $n \geq N$, then $|f_n(x) - f(x)| < \varepsilon$ for all x
(the same N works for all x)

L^2-convergence: $\|f - f_n\|_2 = \left(\int_a^b |f - f_n|^2\,dx\right)^{1/2} \to 0$

Cesaro convergence: $\frac{S_0(x) + S_1(x) + \dots + S_{n-1}(x)}{n} \to f(x)$

pointwise convergence $\Rightarrow$ Cesaro convergence

Uniform convergence $\Rightarrow$ pointwise convergence

Uniform convergence $\Rightarrow$ L^2-convergence on finite intervals.

Nice properties of uniform convergence:

a) If $\{f_n\}$ is a sequence of continuous functions converging uniformly to f, then f is continuous

b) —〃— 〃 —〃—, then $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$

i.e. $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx$ (needs uniform convergence)

Also: If $\{f_n'\}$ converge uniformly to g and $\{f_n\}$ converges at a point c, then $\{f_n\}$ converges uniformly to a differentiable function f and $f' = g$.

$\left(\left(\lim_{n\to\infty} f_n\right)' = \lim_{n\to\infty}(f_n')\right)$

For series: a) If $\sum_{n=0}^{\infty} u_n(x)$ converges uniformly, then

$$\int_a^b \sum_{n=0}^{\infty} u_n(x)\,dx = \sum_{n=0}^{\infty} \int_a^b u_n(x)\,dx$$

b) If $\sum_{n=0}^{\infty} u_n'(x)$ converges uniformly and $\sum_{n=0}^{\infty} u_n(x)$ converges at at least one point, then

$$\left(\sum_{n=0}^{\infty} u_n(x)\right)' = \sum u_n'(x)$$

Weierstrass' M-test: Assume that there is a convergent series $\sum_{n=0}^{\infty} M_n$ of numbers such $|v_n(x)| \leq M_n$ for all $x \in A$ and all $n \in \mathbb{N}$.
Then $\sum_{n=0}^{\infty} v_n(x)$ converges uniformly on A.

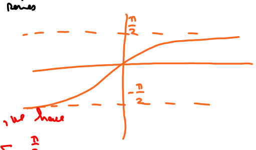
Reminder: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$.

$\sum_{n=1}^{\infty} a^n$ converges when $|a| < 1$.

Problem 1, 2016: a) Show that the series

$$\sum_{n=1}^{\infty} \frac{\arctan(nx)}{n^2}$$

converges uniformly on $\mathbb{R}$.



Solution: Since $|\arctan(nx)| \leq \frac{\pi}{2}$, we have

$$\left|\frac{\arctan(nx)}{n^2}\right| \leq \frac{\frac{\pi}{2}}{n^2} \quad \text{and} \quad \sum \frac{\frac{\pi}{2}}{n^2} \text{ converges.}$$

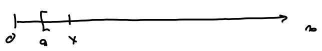
By Weierstrass M-test, the series $\sum \frac{\arctan(nx)}{n^2}$ converge uniformly.

b) Show that f is differentiable for all $x > 0$ and express f' as a series.

Series: $\sum_{n=1}^{\infty} \frac{\arctan(nx)}{n^2}$

Series of Derivatives: $\sum_{n=1}^{\infty} \frac{\frac{1}{1+(nx)^2} \cdot \cancel{n}}{n^{\cancel{2}}} = \sum_{n=1}^{\infty} \frac{1}{n(1+n^2x^2)}$

Choose an interval $[a, \infty)$, $a > 0$, such that $x > a$.



I need to show that the differentiated series $\sum_{n=1}^{\infty} \frac{1}{n(1+n^2x^2)}$ converges uniformly on $[a, \infty)$

$$\frac{1}{n(1+n^2x^2)} \leq \frac{1}{n(1+n^2a^2)} \quad \text{for all } x \in [a, \infty).$$

The series $\sum \frac{1}{n(1+n^2a^2)}$ converges by comparison to $\sum_{n=1}^{\infty} \frac{1}{n^3}$

By Weierstrass, $\sum \frac{1}{n(1+n^2x^2)}$ converges uniformly on $[a, \infty)$

and thus $\left(\sum_{n=1}^{\infty} \frac{\arctan(nx)}{n^2}\right)' = \sum_{n=1}^{\infty} \frac{1}{n(1+n^2x^2)}$

Spaces of bounded functions

$B(X,Y)$ = the set of all bounded functions $f: X \to Y$.

$C_b(X,Y)$ = —— " —— , continuous —— " ——

X compact: $C_b(X,Y) = C(X,Y)$ = the set of all continuous functions $f: X \to Y$

Metric: $\rho(f,g) = \sup\{ d_Y(f(x), g(x)) : x \in X \}$

Theorem: If Y is complete, $B(X,Y)$, $C_b(X,Y)$, $C(X,Y)$ are complete.

Weierstrass' Approximation Theorem: The polynomials are dense in $C([a,b], \mathbb{R})$, i.e. for any continuous function $f: [a,b] \to \mathbb{R}$ there is a sequence $\{p_n\}$ of polynomials converging uniformly to f.

Exam 2018, Problem 5: Assume that $F: C([0,1], \mathbb{R}) \to \mathbb{R}$ is continuous and that $F(p) = 0$ for all polynomials p. Show that $F(f) = 0$ for all $f \in C([0,1], \mathbb{R})$.

Solution: By WAT there is a sequence of polynomial p_n converging uniformly to f, i.e. $p_n \to f$ in $C([0,1], \mathbb{R})$. But then since F is continuous

$$F(f) = F(\lim_{n\to\infty} p_n) \overset{F \text{ is cont.}}{=} \lim_{n\to\infty} F(p_n) = \lim_{n\to\infty} 0 = 0.$$

Chapter 5 Normed Spaces

Def: Assume that $\mathbb{X}$ is vector space over K ($=\mathbb{R}$ or $\mathbb{C}$). A norm on $\mathbb{X}$ is a function $\|\cdot\|: \mathbb{X} \to \mathbb{R}$ such that

(i) $\|\vec{x}\| \geq 0$ with equality iff $\vec{x} = \vec{0}$.

(ii) $\|\alpha \vec{x}\| = |\alpha| \, \|\vec{x}\|$

(iii) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$ is a norm.

Infinite linear combinations: $\vec{u} = \sum_{n=1}^{\infty} \alpha_n \vec{u}_n = \lim \sum_{n=1}^{N} \alpha_n \vec{u}_n$

i.e. $\|\vec{u} - \sum_{n=1}^{N} \alpha_n \vec{u}_n\| \to 0$ as $N \to \infty$.

Basis: $\{u_n\}$ is a basis for $\mathbb{X}$ if any $\vec{u} \in \mathbb{X}$ can be written as a linear combination $\vec{u} = \sum_{n=1}^{\infty} \alpha_n \vec{u}_n$ in exactly one way

(i.e. $\left.\begin{aligned} \vec{u} &= \sum_{n=1}^{\infty} \alpha_n \vec{u}_n \\ \vec{u} &= \sum_{n=1}^{\infty} \beta_n \vec{u}_n \end{aligned}\right\} \Rightarrow \alpha_n = \beta_n.$)

Inner product spaces

Def: $\langle \cdot, \cdot \rangle : \mathbb{X}^2 \to \mathbb{K}$ satisfying

(i) $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$

(ii) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

(iii) $\langle \alpha \vec{u}, \vec{v} \rangle = \alpha \langle \vec{u}, \vec{v} \rangle$

(iv) $\langle \vec{u}, \vec{u} \rangle \geq 0$ with equality iff $\vec{u} = \vec{0}$

Extra: (v) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

(vi) $\langle \vec{u}, \alpha \vec{v} \rangle = \bar{\alpha} \langle \vec{u}, \vec{v} \rangle$

(vii) $\langle \alpha \vec{u}, \alpha \vec{v} \rangle = |\alpha|^2 \langle \vec{u}, \vec{v} \rangle$

Norm: $\|\vec{u}\| = \langle \vec{u}, \vec{u} \rangle^{1/2}$

Metric: $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle^{1/2}$

Pythagorean Th: $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ are orthogonal ($\langle \vec{e}_i, \vec{e}_j \rangle = 0$ when $i \neq j$), then $\|\vec{e}_1 + \vec{e}_2 + \cdots + \vec{e}_n\|^2 = \|\vec{e}_1\|^2 + \cdots + \|\vec{e}_n\|^2$

Schwarz' inequality: $|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \|\vec{v}\|$.

Bessel's inequality: If $\{\vec{e}_n\}_{n \in \mathbb{N}}$ is an orthonormal set, then

$$\sum_{n=1}^{\infty} |\underbrace{\langle \vec{u}, \vec{e}_n \rangle}_{\text{Fourier coefficients}}|^2 \leq \|\vec{u}\|^2 \text{ for all } \vec{u}$$

Parseval's Theorem: If $\{\vec{e}_n\}$ is an orthonormal basis, then

$$\sum_{n=1}^{\infty} |\langle \vec{u}, \vec{e}_n \rangle|^2 = \|\vec{u}\|^2$$

and

$$\vec{u} = \sum_{n=1}^{\infty} \langle \vec{u}, \vec{e}_n \rangle \vec{e}_n.$$

Linear maps

$A: \mathbb{X} \to \mathbb{Y}$ normed spaces

Linear: $A(\alpha \vec{u} + \beta \vec{v}) = \alpha A\vec{u} + \beta A\vec{v}$

Boundedness: $\|A\vec{u}\| \leq M \|\vec{u}\|$ for some $M \in \mathbb{R}$.

Operator norm: $\|A\| = \sup \left\{ \frac{\|A\vec{u}\|}{\|\vec{u}\|} : \vec{u} \neq 0 \right\}$

Inverse: A is invertible if it is bijective and the inverse function is $\underbrace{\text{linear}}_{\text{automatic}}$ and $\underbrace{\text{bounded}}_{\text{not automatic.}}$