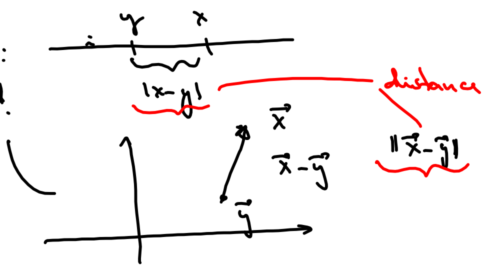
Chapter 1

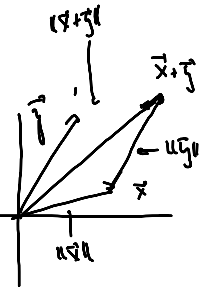
Convergence
Continuity
Completeness
Compactness

} in the context of $\mathbb{R}^d$ $\Longrightarrow$ context of metric spaces

Distance / in $\mathbb{R}$:
\ in $\mathbb{R}^d$:



Triangle inequality / in $\mathbb{R}$ $\quad |x+y| \le |x| + |y|$
\ in $\mathbb{R}^d$ $\quad \|\vec{x}+\vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$



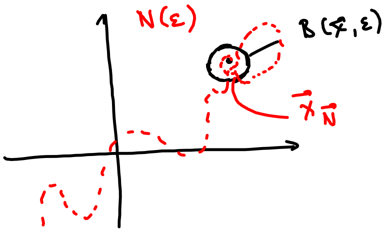
Convergence

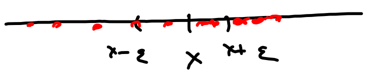
In $\mathbb{R}$: $\quad x_n \to x$

In $\mathbb{R}^d$: $\quad \vec{x}_n \to \vec{x}$

Definition: The sequence $\{\vec{x}_n\}$ converges to $\vec{x}$ if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that if $n \ge N$, then $|\vec{x}_n - \vec{x}| < \varepsilon$

$N(\varepsilon)$





Proposition: If $\{\vec{x}_n\}$ converges to $\vec{x}$ and $\{\vec{y}_n\}$ converges to $\vec{y}$, then $\{x_n + y_n\}$ converges to $x+y$.

Proof: Given an $\varepsilon > 0$, we must prove that there is an $N \in \mathbb{N}$ such that when $n \ge N$, then $\|(x_n+y_n) - (x+y)\| < \varepsilon$.

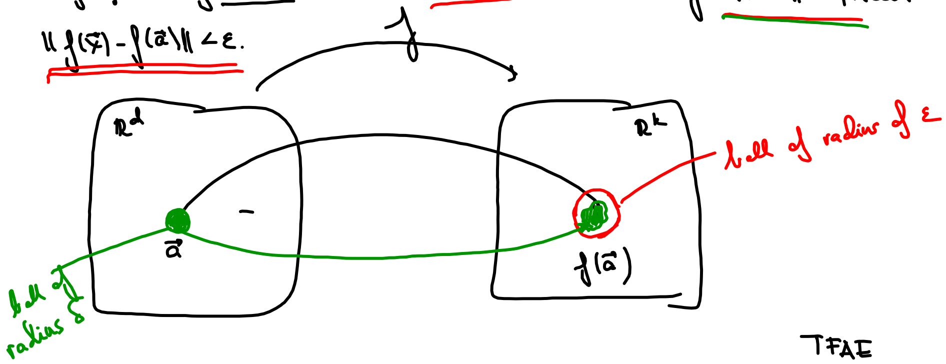
Thinking: $\|(\vec{x}_n+\vec{y}_n) - (\vec{x}+\vec{y})\| = \|(\vec{x}_n - \vec{x}) + (\vec{y}_n - \vec{y})\| \le \underbrace{\|\vec{x}-\vec{x}_n\|}_{\varepsilon/2} + \underbrace{\|\vec{y}-\vec{y}_n\|}_{\varepsilon/2}$

Since $\vec{x}_n \to \vec{x}$ and $\vec{y}_n \to \vec{y}$, I can get $\|\vec{x}_n - \vec{x}\| < \frac{\varepsilon}{2}$ and $\|\vec{y}_n - \vec{y}\| < \frac{\varepsilon}{2}$ by choosing n large enough

Since $\vec{x}_n \to \vec{x}$, there is an $\overset{N_1(\frac{\varepsilon}{2})}{N_1} \in \mathbb{N}$ such that if $n \ge N_1$, then $\|\vec{x}_n - \vec{x}\| < \frac{\varepsilon}{2}$. Since $\vec{y}_n \to \vec{y}$, there is an N_2 such that if $n \ge N_2$, then $\|\vec{y}_n - \vec{y}\| < \frac{\varepsilon}{2}$. Choose $N = \max\{N_1, N_2\}$; then if $n \ge N$, then $n \ge N_1$ and $n \ge N_2$. Hence if $n \ge N$, then
$\|(x_n+y_n) - (x+y)\| \le \|\vec{x} - \vec{x}_n\| + \|\vec{y} - \vec{y}_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Continuity

Definition: A function $f:\mathbb{R}^d \to \mathbb{R}^k$ is continuous at a point $\vec{a} \in \mathbb{R}^d$ if for every $\varepsilon > 0$, there is $\delta > 0$ such that if $\|\vec{x}-\vec{a}\| < \delta$, then $\|f(\vec{x}) - f(\vec{a})\| < \varepsilon$.



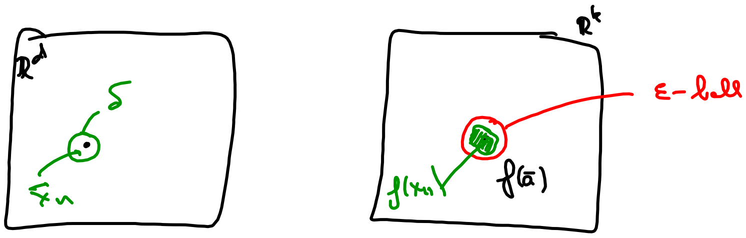
TFAE

Theorem: Let $f:\mathbb{R}^d \to \mathbb{R}^k$ and let $\vec{a} \in \mathbb{R}^d$. The following are equivalent

(i) f is continuous at $\vec{a}$

(ii) $f(\vec{x}_n) \to f(\vec{a})$ for all sequences $\vec{x}_n \to \vec{a}$.

Proof: (i) $\Rightarrow$ (ii) Assume f continuous at $\vec{a}$ and that $\vec{x}_n \to \vec{a}$. We need to prove that $f(\vec{x}_n) \to f(\vec{a})$. Given $\varepsilon > 0$, we need show that there always is an $N \in \mathbb{N}$ such that if $n \geq N$, then $|f(\vec{x}_n) - f(\vec{a})| < \varepsilon$.



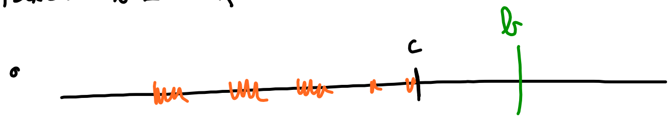
Since f is cont. at $\vec{a}$, there is a $\delta > 0$ such that if $|\vec{x} - \vec{a}| < \delta$, then $\|f(\vec{x}) - f(\vec{a})\| < \varepsilon$. Since $\vec{x}_n \to \vec{a}$, there is an $N \in \mathbb{N}$ such that if $n \geq N$, then $|\vec{x}_n - \vec{a}| < \delta$. This means $N(\delta)$ that if $n \geq N$, then $\|f(\vec{x}_n) - f(\vec{a})\| < \varepsilon$

(ii) $\Rightarrow$ (i) (done contrapositively; i.e. $\neg$(i) $\Rightarrow \neg$(ii))

Assume that f is not continuous at $\vec{a}$. This means that there is an $\varepsilon > 0$ such that no matter which $\delta > 0$ I choose, there is a point $\vec{x}$ such $|\vec{x} - \vec{a}| < \delta$, but $\|f(\vec{x}) - f(\vec{a})\| \geq \varepsilon$. For $\delta = \frac{1}{n}$, this means that I can find $\vec{x}_n$ such that $|\vec{x}_n - a| < \frac{1}{n}$ and $\|f(\vec{x}_n) - f(\vec{a})\| \geq \varepsilon$. The sequence $\{\vec{x}_n\}$ clearly converges to $\vec{a}$, but $\{f(\vec{x}_n)\}$ does not converge to $f(\vec{a})$. This finishes the proof.

Completeness of $\mathbb{R}$

A subset A of $\mathbb{R}$ is upward bounded if there is a number b such $b \geq a$ for all $a \in A$. We then call b an upper bound for A.



c is called the supremum (or least upper bound) of A if is an upper bound for A and smaller that all other upper bounds. We denote $c = \sup A$

Completeness principle: Every nonempty upward bounded set $A \subseteq \mathbb{R}$ has a supremum.

Example: The completeness principle does not hold in $\mathbb{Q}$

$$A = \{ q \in \mathbb{Q} : q^2 < 2 \}$$

There is no rational number that is the supremum of A (in $\mathbb{R}$: $\sup A = \sqrt{2}$).

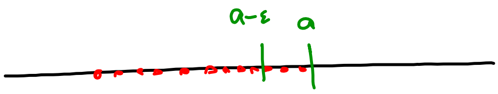
Proposition: Every $\underbrace{\text{bounded}}_{\substack{|x_n| \leq M \\ \text{for all } n}}$, $\underbrace{\text{monotone}}_{\substack{\text{increasing} \\ \text{or decreasing}}}$ sequence in $\mathbb{R}$ converges

Proof (for $\{x_n\}$ increasing): The set

$$A = \{ x_n : n \in \mathbb{N} \}$$

is nonempty and bounded, and hence has a least upper bound a,

$$a = \sup A.$$

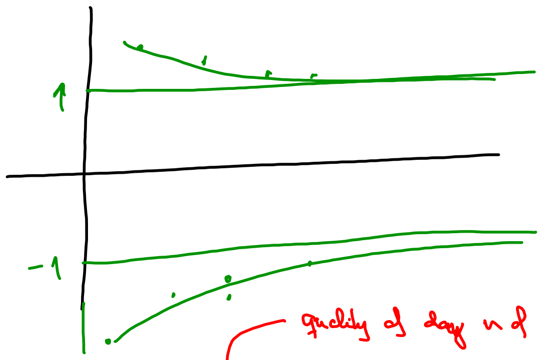


We want to prove that $\{x_n\}$ converges to a. Given $\varepsilon > 0$, we must find an $N \in \mathbb{N}$ such that if $n \geq N$, then $|x_n - a| < \varepsilon$. Since a is the least upper bound for A, $a - \varepsilon$ is not an upper bound, so there must be an $x_N \in A$ such that $x_N > a - \varepsilon$. Since the sequence is increasing, this means that $\underbrace{a - \varepsilon < x_n \leq a}_{|x_n - a| < \varepsilon}$ for all $n \geq N$.

This completes the proof.

liminf and limsup

Example: $x_n = (-1)^n\left(1+\frac{1}{n}\right)$, $n = 1, 2, 3, \ldots$



quality of day n of your life.

Given a sequence $\{x_n\}$ of real numbers define:

might be ∞

$M_n = \sup\{x_k : k \geq n\}$ the top quality of the rest of your days

$m_n = \inf\{x_k : k \geq n\}$ the lowest quality of the rest of your days

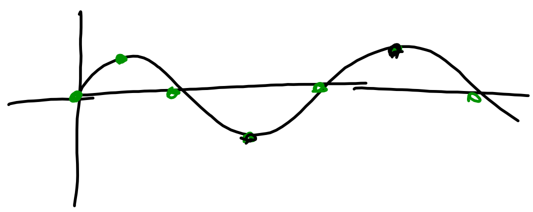
might be $-\infty$.

M_n is decreasing (possibly to $-\infty$)
m_n is increasing (possibly to ∞)
always have limits (allowing $\pm\infty$)

$$\limsup x_n = \lim_{n\to\infty} M_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\}$$

$$\liminf x_n = \lim_{n\to\infty} m_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\}$$

Example: $x_n = \sin\left(n\frac{\pi}{2}\right)$



$\limsup x_n = 1$

$\liminf x_n = -1$

Proposition:

$$\lim_{n\to\infty} x_n = a \iff \limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = a$$

Proof: Assume that $\underbrace{\limsup a_n}_{\lim_{n\to\infty} M_n} = \underbrace{\liminf}_{\lim_{n\to\infty} m_n} = a$

$$m_n \leq x_n \leq M_n$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$a \quad a \quad a$$