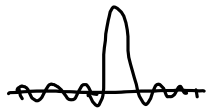
Aim: $s_n(x) \to f(x)$ conditions

$$s_n(x) = \sum_{k=-n}^{n} \alpha_k e_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u) D_n(u)\, du$$



Riemann-Lebesgue Lemma: If $f \in D$, then $\alpha_n \to 0$ as $n \to \pm\infty$.

Proof: By Parseval's inequality

$$\sum_{n=-\infty}^{\infty} |\alpha_n|^2 \le \|f\|_2 < \infty, \text{ hence } \alpha_n \to 0.$$

Corollary: If $f \in D$ and $[a,b] \subseteq [-\pi,\pi]$, then

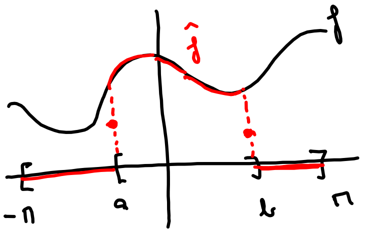
$$\int_a^b f(x)\cos nx \to 0 \quad \text{as } n \to \infty$$

and
$$\int_a^b f(x)\sin nx \to 0 \quad \text{as } n \to \infty$$

Proof: Note that $\int_a^b f(x)\underbrace{e^{-inx}}\, dx = \int_a^b f(x)\cos nx\, dx - i\int_a^b f(x)\sin nx\, dx$,

and hence it suffices to prove that $\int_a^b f(x) e^{-inx} \to 0$.

Let $\hat{f}$ be the function:



Then $\frac{1}{2\pi}\int_a^b f(x)\, e^{-inx}\, dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(x) e^{-inx}\, dx$$

$$= \hat{\alpha}_n \to 0 \quad \text{as } n \to \pm\infty$$

Fourier coefficient of $\hat{f} \in D$.

Corollary: If $f \in D$ and $[a,b] \subseteq [-\pi,\pi]$, then

(i) $\int_a^b f(x)\cos[(n+\tfrac{1}{2})x]\, dx \to 0$ as $n \to \infty$

(ii) $\int_a^b f(x)\underline{\sin[(n+\tfrac{1}{2})x]\, dx} \to 0$ — " —

$$\boxed{\sin(u+v) = \sin u \cos v + \cos u \sin v}$$

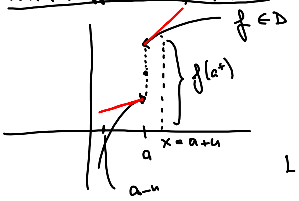
Proof (ii): $\int_a^b f(x)\sin[\underbrace{(n+\tfrac{1}{2})x}]\, dx = \int_a^b f(x)\sin[nx + \tfrac{x}{2}]\, dx$

$$= \int_a^b f(x)\,[\underbrace{\sin nx \cos \tfrac{x}{2}} + \cos nx \sin\tfrac{x}{2}]\, dx$$

$$= \int_a^b \underbrace{f(x)\cos\tfrac{x}{2}}_{D} \sin nx\, dx + \int \underbrace{f(x)\sin\tfrac{x}{2}}_{D} \cos nx\, dx \to 0.$$

Pointwise convergence of Fourier series (10.6)

What happens at jump points:



Define

$$f'_+(a) = \lim_{u\to 0^+} \frac{f(a+u) - f(a^+)}{u} \quad \text{I}$$

$$f'_-(a) = \lim_{u\to 0^+} \frac{f(a-u) - f(a^-)}{-u} \quad \text{II}$$

Let us look at

$$\lim_{u\to 0} \frac{f(a+u)+f(a-u)-2f(a)}{u} = \lim_{u\to 0} \frac{f(a+u)+f(a-u) - f(a^+) - f(a^-)}{u}$$

$$= \lim_{u\to 0} \frac{f(a+u) - f(a^+)}{u} + \lim_{u\to 0} \frac{f(a-u) - f(a^-)}{u}$$

$$= f'_+(a) - f'_-(a).$$

Want to find conditions such that $s_n(x) \to f(x)$.

Need to show that

$$|s_n(x) - f(x)| \to 0.$$

Recall that

$$s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)D_n(u)\,du, \quad D_n(u) = \frac{\sin[(n+\frac{1}{2})u]}{\sin\frac{u}{2}}$$

Substituting $t = -u$:

$$s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+u)D_n(u)\,du$$

also

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)D_n\,du = f(x)\Big(\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(u)\,du}_{1}\Big)$$

$$|s_n(x) - f(x)| = \Big|\frac{1}{4\pi}\int_{-\pi}^{\pi} f(x-u)D_n(u)\,du$$
$$+ \frac{1}{4\pi}\int_{-\pi}^{\pi} f(x+u)D_n(u)\,du$$
$$- \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)D_n(u)\,du\Big|$$

$$= \frac{1}{4\pi}\Big|\int_{-\pi}^{\pi} [f(x+u) + f(x-u) - 2f(x)]D_n(u)\,du\Big|$$

Lemma: Assume that there is $\eta > 0$ such that

$$\frac{1}{4\pi}\int_{-\eta}^{\eta} [f(x+u)+f(x-u)-2f(x)]D_n(u)\,du \to 0$$

then $s_n(x) \to f(x)$.

Proof: Must prove that the following integral converges to 0:

$$\frac{1}{4\pi}\int_{-\pi}^{\pi} [f(x+u)+f(x-u)-2f(x)]D_n(u)\,du$$

$$= \frac{1}{4\pi}\Big[\underbrace{\int_{-\pi}^{-\eta} \cdots}_{\downarrow\ 0} + \underbrace{\int_{-\eta}^{\eta} \cdots}_{\downarrow\ 0} + \underbrace{\int_{\eta}^{\pi}}_{\downarrow\ 0}\Big]$$

Look at

$$\int_{\eta}^{\pi} \underbrace{[f(x+u)+f(x-u)-2f(x)]\frac{1}{\sin\frac{u}{2}}}_{D} \sin[(n+\tfrac{1}{2})u]\,du \to 0$$

by corollary.

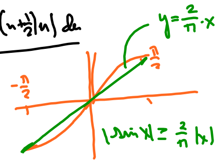
Same for $\int_{-\pi}^{-\eta}$.

Dini's Test: Assume that $f \in D$ and that there is a $\delta > 0$ such that
$$\int_{-\delta}^{\delta} \left| \frac{f(x+u) + f(x-u) - 2f(x)}{u} \right| du < \infty.$$
Then $s_n(x) \to f(x)$.

Proof: By the lemma it suffices to prove that
$$\frac{1}{4\pi}\int_{-\delta}^{\delta} [f(x+u) + f(x-u) - 2f(x)]\, D_n(u)\, du \to 0 \quad \text{as } n \to \infty.$$
Given an $\varepsilon > 0$, we have to show that
$$\frac{1}{4\pi}\left|\int_{-\delta}^{\delta} [f(x+u) + f(x-u) - 2f(x)] D_n(u)\, du\right| < \varepsilon \quad \text{for all sufficiently large } n.$$
Since $\int_{-\delta}^{\delta} \left|\frac{f(x+u)+f(x-u)-2f(x)}{u}\right| du < \infty$, we can find an $\eta > 0$ such that
$$\int_{-\eta}^{\eta} \left|\frac{f(x+u) + f(x-u) - 2f(x)}{u}\right| du < \varepsilon$$
Hence

$$D_n(u) = \frac{\sin(n+\frac{1}{2})u}{\sin\frac{u}{2}}$$

$$\left| \frac{1}{4\pi}\int_{-\eta}^{\eta} [f(x+u) + f(x-u) - 2f(x)] D_n(u)\, du \right|$$
$$\le \frac{1}{4\pi}\int_{-\eta}^{\eta} \frac{|f(x+u) + f(x-u) - 2f(x)|}{|\sin\frac{u}{2}|} \underbrace{|\sin(n+\tfrac{1}{2})u|}_{\le 1} du$$
$$\le \frac{1}{4\cancel{\pi}}\int_{-\eta}^{\eta} \frac{|f(x+u) + f(x-u) - 2f(x)|\,\cancel{\pi}}{|u|} du$$
$$= \frac{1}{4}\int_{-\eta}^{\eta} \left|\frac{f(x+u) + f(x-u) - 2f(x)}{u}\right| du < \frac{\varepsilon}{4}$$

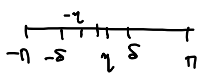


$|\sin x| \ge \frac{2}{\pi}|x|$

$|\sin\frac{u}{2}| \ge \frac{2}{\pi}\left|\frac{u}{2}\right| = \frac{|u|}{\pi}$

$\frac{1}{|\sin\frac{u}{2}|} \le \frac{\pi}{|u|}$

Thus
$$\frac{1}{4\pi}\left|\int_{-\delta}^{\delta} [f(x+u) + f(x-u) - 2f(x)] D_n(u)\, du\right|$$



$$= \frac{1}{4\pi}\left| \underbrace{\int_{-\delta}^{-\eta} \cdots}_{\le \varepsilon/4} + \underbrace{\int_{-\eta}^{\eta} \cdots}_{\le \varepsilon/4} + \underbrace{\int_{\eta}^{\delta} \cdots}_{\le \varepsilon/4} \right| < \varepsilon$$

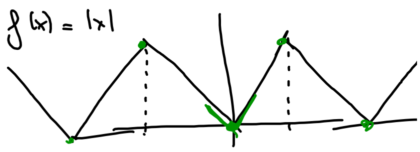
Let's take a look at
$$\frac{1}{4\pi}\int_{\eta}^{\delta} [f(x+u) + f(x-u) - 2f(x)] D_n(u)\, du$$
$$= \frac{1}{4\pi}\int_{\eta}^{\delta} \underbrace{\frac{[f(x+u) + f(x-u) - 2f(x)]}{\sin\frac{u}{2}}}_{\in D} \sin[(n+\tfrac{1}{2})u]\, du \to 0$$

QED

Condition: $\int_{-\delta}^{\delta} \left|\frac{f(x+u) + f(x-u) - 2f(x)}{u}\right| du < \infty$

satisfied if $f_+'(x)$ and $f_-'(x)$ exists

Mandatory Assignment: $f(x) = |x|$



One-sided derivatives exist $\Rightarrow$

Fourier series converges everywhere.

Section 5.4 Linear operators.

Def: Assume that U, V are linear spaces and that $A: U \to V$. Then A is called a linear operator/map if
(i) $A(\alpha \vec{u}) = \alpha A(\vec{u})$ for all $\alpha \in \mathbb{K}$ and $\vec{u} \in U$.
(ii) $A(\vec{u} + \vec{v}) = A(\vec{u}) + A(\vec{v})$ for all $u, v \in U$.

Prop: If A is linear, then
$$A(\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \cdots + \alpha_n \vec{u}_n) = \alpha_1 A_1(\vec{u}_1) + \alpha_2 A(\vec{u}_2) + \cdots \alpha_n A(\vec{u}_n)$$

Def: Assume that $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_V)$ are two normed linear spaces. Then ~~a~~ linear operator $A: U \to V$ is called bounded if there is a constant K such that
$$\|A\vec{u}\|_V \leq K \|\vec{u}\|_U$$

Tricky terminology: Then A is a bounded operator does not mean that $\|A(\alpha \vec{u})\|_V = \cancel{\text{bounded}}$ function.

$V = |\alpha| \|A(\vec{u})\|_V$

Lemma: If A is a bounded operator, then A is uniformly continuous.

Proof: $\|A(\vec{u}) - A(\vec{v})\|_V = \|A(\vec{u} - \vec{v})\|_V \leq K \|\vec{u} - \vec{v}\|_U$

Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{K}$ and we see that if $\|u - v\|_U < \delta$, that $\|A(\vec{u}) - A(\vec{v})\| \leq K\|\vec{u} - \vec{v}\| < K \cdot \frac{\varepsilon}{K} = \varepsilon$.

Lemma: If A is continuous at 0, then A is bounded.

Proof: Contrapositive: Assume that A is not bounded, then for any $n \in \mathbb{N}$, there is an element $\vec{u}_n \in U$ such that
$$\|A(\vec{u}_n)\|_V \geq n \|\vec{u}_n\|_U$$

Put $\vec{v}_n = \frac{\vec{u}_n}{n \|\vec{u}_n\|_U}$. Then $\|\vec{v}_n\|_U = \frac{1}{n}$, and hence $\vec{v}_n \to \vec{0}$.

But $A(\vec{v}_n) = A\left(\frac{\vec{u}_n}{n\|\vec{u}_n\|}\right) = \frac{1}{n\|\vec{u}_n\|} A(\vec{u}_n)$

and
$$\|A(\vec{v}_n)\|_V = \frac{1}{n\|\vec{u}_n\|} \|A\vec{u}_n\|_V \geq 1$$

Hence $A(\vec{v}_n) \not\to \vec{0} = A\vec{0}$

Thus $\vec{v}_n \to 0$, but $A(\vec{v}_n) \not\to A(\vec{0})$. (hence A is not continuous)

f cont means $x_n \to x$, then $f(x_n) \to f(x)$