Bounded linear operators

Last time:

Theorem: For a linear operator $A: U \to V$ the following are equivalent

(i) A is bounded

(ii) A is uniformly continuous

(iii) A is continuous at the origin.

Convention: $A(\vec{u}) = A\vec{u}$

A bounded means $\|A\vec{u}\|_V \le K\|\vec{u}\|_U$

$$\frac{\|A\vec{u}\|_V}{\|\vec{u}\|_U} \le K.$$

Definition: The operator norm $\|A\|$ of $A: U \to V$ is defined by

$$\|A\| = \sup\left\{\frac{\|A\vec{u}\|_V}{\|\vec{u}\|_U} : \vec{u} \neq 0\right\} = \sup\left\{\|A\vec{u}\|_V : \|\vec{u}\| = 1\right\}$$

Proposition: Assume that $A: U \to V$ and $B: V \to W$ are bounded linear operators. Then $B \circ A = BA : U \to W$ is a bounded linear operator and $\|BA\| \le \|B\|\|A\|$

Proof. Linearity:

$$(BA)(\alpha\vec{u} + \beta\vec{v}) = B(A(\alpha\vec{u} + \beta\vec{v})) = B(\alpha A\vec{u} + \beta A\vec{v})$$

$$= \alpha B(A\vec{u}) + \beta B(A\vec{v}) = \alpha BA(\vec{u}) + \beta BA(\vec{v})$$

Bounded: For any $\vec{u} \in U$, we have

$$\|BA(\vec{u})\|_W = \|B(A\vec{u})\|_W \le \|B\|\|A\vec{u}\|_V \le \|B\|\|A\|\|\vec{u}\|_U$$

hence

$$\frac{\|BA(\vec{u})\|_W}{\|u\|_U} \le \|B\|\|A\|$$

Taking sup, we get

$$\|BA\| = \sup\left\{\frac{\|BA(\vec{u})\|_W}{\|u\|_U} : \vec{u} \neq 0\right\} \le \|B\|\|A\|$$

Definition: $\mathcal{L}(U,V)$ is the set of all bounded linear operators from U to V.

Proposition: $\mathcal{L}(U,V)$ is a linear space.

Proof. Need to define two operations

Multiplication by scalar: αA : $(\alpha A)(\vec{u}) = \alpha(A\vec{u})$

Addition: $A+B$: $(A+B)(\vec{u}) = A\vec{u} + B\vec{u}$

Need to check all the axioms for a linear space

Good luck!

Proposition: The operator norm is a norm on $\mathcal{L}(U,V)$. Hence $(\mathcal{L}(U,V), \|\cdot\|)$ is a normed space.

Proof: There are three properties to check:

(i) $\|A\| \ge 0$ with equality only if $A = 0$

Check: $\|A\| = \sup\left\{\underbrace{\frac{\|A\vec{u}\|_V}{\|\vec{u}\|_U}}_{\ge 0} : \|\vec{u}\| \neq 0\right\} \ge 0$

If $A \neq 0$, there is a $u \in U$ such that $A\vec{u} \neq 0$, and hence $\|A\vec{u}\|_V > 0$. But then

$$\|A\| \ge \frac{\|A\vec{u}\|_V}{\|\vec{u}\|_U} > 0.$$

(ii) $\|\alpha A\| = |\alpha|\|A\|$:

$$\|\alpha A\| = \sup\left\{\frac{\|\alpha A(\vec{u})\|_V}{\|\vec{u}\|_U} : \vec{u} \neq 0\right\} = \sup\left\{\frac{|\alpha|\|A\vec{u}\|_V}{\|\vec{u}\|_U} : \vec{u} \neq 0\right\}$$

$$= |\alpha| \underbrace{\sup\left\{\frac{\|A\vec{u}\|_V}{\|\vec{u}\|_U} : \vec{u} \neq 0\right\}}_{\|A\|} = |\alpha|\|A\|.$$

(iii) $\|A+B\| \le \|A\| + \|B\|$:

$$\|A+B\| = \sup\left\{\frac{\|(A+B)\vec{u}\|_V}{\|\vec{u}\|_U} : \|\vec{u}\| \neq 0\right\}$$

$$\le \sup\left\{\frac{\|A\vec{u}\|_V + \|B\vec{u}\|_V}{\|\vec{u}\|_U} : \|\vec{u}\| \neq 0\right\}$$

$$= \sup\left\{\frac{\|A\vec{u}\|_V}{\|\vec{u}\|_U} + \frac{\|B\vec{u}\|_V}{\|\vec{u}\|_U} : \|\vec{u}\| \neq 0\right\}$$

$$\le \sup\left\{\frac{\|A\vec{u}\|_V}{\|\vec{u}\|_U} : \|\vec{u}\| \neq 0\right\} + \sup\left\{\frac{\|B\vec{u}\|_V}{\|\vec{u}\|_U} : \|\vec{u}\| \neq 0\right\}$$

$$= \|A\| + \|B\|.$$

Theorem: If U and V are normed spaces and V is complete, then $\mathcal{L}(U,V)$ is complete in the operator norm.

Proof: Assume that $\{A_n\}$ is a Cauchy sequence in $\mathcal{L}(U,V)$. We must prove that $\{A_n\}$ converges to an $A \in \mathcal{L}(U,V)$.

Steps: (i) Define A
(ii) Show that A is linear
(iii) Show that A is bounded
(iv) Show that $A_n \to A$ in the operator norm.

Step (i): Idea: $A\vec{u} = \lim_{n\to\infty} A_n\vec{u}$ (need to show that limit exists)

Since V is complete, it suffices to show that $\{A_n\vec{u}\}$ is a Cauchy sequence:

$$\|A_m\vec{u} - A_n\vec{u}\| = \|(A_m - A_n)\vec{u}\| \le \underbrace{\|A_m - A_n\|}_{\text{can get as small as I want by choosing } m,n \text{ large enough}}\|u\|$$

Hence $\{A_n\vec{u}\}$ converges, and we may define

$$A\vec{u} = \lim_{n\to\infty} A_n\vec{u}.$$

(ii) Must show that A is linear:

$$A(\alpha\vec{u}+\beta\vec{v}) = \lim_{n\to\infty} A_n(\alpha\vec{u}+\beta\vec{v}) \overset{A_n \text{ is linear}}{=} \lim_{n\to\infty}(\alpha A_n\vec{u} + \beta A_n\vec{v})$$

$$\alpha \underbrace{\lim A_n\vec{u}}_{A\vec{u}} + \beta \underbrace{\lim A_n\vec{v}}_{A\vec{v}} = \alpha A\vec{u} + \beta A\vec{v}.$$

(iii) By the inverse triangle inequality

$$\Big|\|A_n\| - \|A_m\|\Big| \le \underbrace{\|A_n - A_m\|}_{\text{Cauchy sequence}}$$

and hence $\{\|A_n\|\}$ is a Cauchy sequence and thus bounded.
Let $K \ge \|A_n\|$ for all n:

$$\|A\vec{u}\|_V = \|\lim_{n\to\infty} A_n\vec{u}\| = \lim_{n\to\infty} \underbrace{\|A_n u\|_V}_{\|A_n\|\|u\| \le K\|u\|} \le K\|u\|_U$$

which shows that A is bounded.

(iv) We must show that $A_n \to A$, hence given $\varepsilon > 0$, we must find an $N \in \mathbb{N}$ such that $\|A - A_n\| < \varepsilon$ when $n \ge N$. Since $\{A_n\}$ is a Cauchy sequence, I can find an N such that when $m,n \ge N$, then $\|A_m - A_n\| < \varepsilon/2$. This means that for any $\vec{u} \in U$,

$$\|A\vec{u} - A_n\vec{u}\|_V = \lim_{m\to\infty} \|A_m\vec{u} - A_n\vec{u}\|_V \le \lim_{m\to\infty} \|(A_m - A_n)\vec{u}\|_V$$

$$\le \lim_{m\to\infty} \|A_m - A_n\|\|u\|_V < \frac{\varepsilon}{2}\|\vec{u}\|$$

Hence for all $\vec{u}$

$$\frac{\|(A - A_n)\vec{u}\|_V}{\|\vec{u}\|_U} < \frac{\varepsilon}{2}$$

and thus

$$\|A - A_n\| = \sup\left\{\frac{\|(A-A_n)\vec{u}\|_V}{\|\vec{u}\|_U} : \vec{u} \ne 0\right\} \le \frac{\varepsilon}{2} < \varepsilon$$

Inverse operators

If $A: U \to V$ is a bijective, linear operator, what kind of inverse function does it have.

Proposition: The inverse function $B: V \to U$ is linear.

Proof: We need to prove that $B(\alpha\vec{u}+\beta\vec{v}) = \alpha B\vec{u} + \beta B\vec{v}$

We have

$$A(\underline{B(\alpha\vec{u}+\beta\vec{v})}) = \alpha\vec{u}+\beta\vec{v} \quad \text{(since } A \text{ and } B \text{ are inverses)}$$

$$A(\underline{\alpha B\vec{u}+\beta B\vec{v}}) = \alpha \underbrace{A(B\vec{u})}_{\vec{u}} + \beta \underbrace{A(B\vec{v})}_{\vec{v}} = \alpha\vec{u}+\beta\vec{v}.$$

Since A is injective, this means that $B(\alpha\vec{u}+\beta\vec{v}) = \alpha B\vec{u}+\beta B\vec{v}$.

Remark: Unfortunately, the inverse of a bounded, linear map need not be bounded.

Idea: $\{e_n\}$ $\quad A e_n = \frac{e_n}{n}$

$$B(\underline{e_n}) = \underline{n e_n}$$

Bounded Inverse Theorem: If U, V are complete, then a bounded, bijective, linear operator $A: U \to V$ has a bounded inverse.

Definition: A bounded linear operator $A: U \to V$ is called invertible if it has a bounded, linear inverse $B: V \to U$. This inverse is then denoted by A^{-1}.

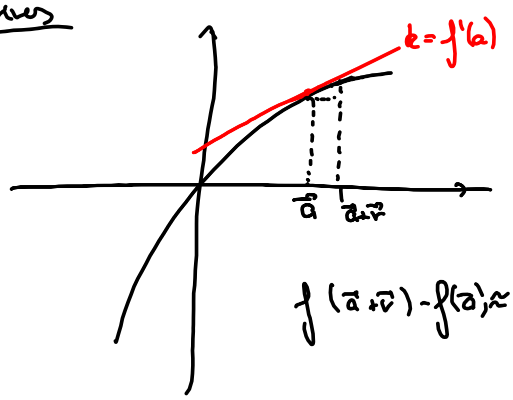
Proposition: If $A: U \to V$ and $B: V \to W$ are invertible, then BA is invertible and $(BA)^{-1} = A^{-1}B^{-1}$.

Banach's lemma: Assume that U is a complete normed space and that $C, D: U \to U$ are bounded, linear operators. If C is invertible and $\|C-D\| < \frac{1}{\|C^{-1}\|}$, then D is also invertible. ("if you're close enough to something invertible you are yourself invertible")

Derivatives

What are they?

In one dimension:



$f(\vec{a}+\vec{r})-f(\vec{a}) \approx f'(a)r$ ← linear approx

In higher dimensions $\vec{F}:\mathbb{R}^n \to \mathbb{R}^m$

Jacobi matrix: $\vec{F}'(\vec{a})$

$\vec{F}(\vec{a}+\vec{r}) - \vec{F}(\vec{a}) \approx \underline{\vec{F}'(\vec{a})\vec{r}}$

linear approx,

New situation: $\vec{F}: \vec{U} \to \vec{V}$

What is the derivative at $\vec{a}$? $\underbrace{\text{Linear approximations}}_{A:U\to V}$ to $\underline{\vec{F}(\vec{a}+\vec{r})-\vec{F}(\vec{a})} \approx A(\vec{r})$

Definition: Assume that we have a function $\vec{F}: U \to V$. A derivative of $\vec{F}$ at $\vec{a}$ is a bounded, linear operator $A: U \to V$ such that (informally: $\underline{\vec{F}(\vec{a}+\vec{r}) - \vec{F}(\vec{a}) \approx A(\vec{r})}$)

$$\sigma(\vec{r}) = \vec{F}(\vec{a}+\vec{r}) - \vec{F}(\vec{a}) - A(\vec{r})$$

goes to 0 faster than $\vec{r}$, i.e.

$$\lim_{r \to 0} \frac{\sigma(r)}{\|\vec{r}\|} = 0$$