Chap 3 $\downarrow$ Metric space

Chap 4 $\downarrow$ cont / conv

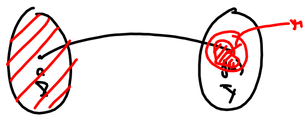
Metric spaces of functions.

Metric spaces of bounded functions

(X, d_X), (Y, d_Y) metric space

$f: X \to Y$ (bounded)

Def. A function $f: X \to Y$ is bounded if there is an $a \in X$ and an $M \in \mathbb{R}$ such that for all $x \in X$
$d_Y(f(x), f(a)) \leq M$. (If this works for some $a \in X$, it works for all a)



Lemma: If $f, g: X \to Y$ are bounded, then there is $K \in \mathbb{R}$ such that

$$d_Y(f(x), g(x)) \leq K \text{ for all } x \in X.$$

Proof: Fix a point $a \in X$. Then

$$d_Y(f(x), g(x)) \leq \underbrace{d_Y(f(x), f(a))} + d_Y(f(a), g(a)) + \underline{d_Y(g(a), g(x))}$$

$$\leq M_f + d_Y(f(a), g(a)) + M_g = K.$$

Let

$$B(X,Y) = \{f: X \to Y : f \text{ is bounded}\}$$

For $f, g \in B(X,Y)$, define

$$\rho(f,g) = \sup\{\underline{d_Y(f(x), g(x))} : x \in X\}$$

By the lemma, $\rho(f,g) < \infty$.

Theorem: ρ is a metric on $B(X,Y)$

Proof: Since $\rho: B(X,Y) \times B(X,Y) \to [0, \infty)$, we only need to check that ρ satisfies the three conditions for being a metric.

(i) Positivity: $\rho(f,g) \geq 0$ with equality if and only if $f = g$.

Check: $\rho(f,g) = \sup\{\underbrace{d_Y(f(x), g(x))}_{\geq 0 \text{ with equality iff } f(x) = g(x)} : x \in X\} \geq 0$

with equality if and only if $d_Y(f(x), g(x)) = 0$ for all x, i.e. if $f(x) = g(x)$ for all x, hence only if $f = g$.

(ii) Symmetry: $\rho(f,g) = \rho(g,f)$

Check: $\rho(f,g) = \sup\{\underbrace{d_Y(f(x), g(x))} : x \in X\}$

$$= \sup\{d_Y(g(x), f(x)) : x \in X\}$$

$$= \rho(g,f).$$

(iii) Triangle inequality: $\rho(f,g) \leq \rho(f,h) + \rho(h,g)$

$$d_Y(f(x), g(x)) \leq d_Y(f(x), h(x)) + d_Y(h(x), g(x))$$

$$\leq \rho(f,h) + \rho(h,g)$$

for all $x \in X$. Taking the supremum over all $x \in X$, we get

$$\rho(f,g) = \sup\{d_Y(f(x), g(x)) : x \in X\} \leq \rho(f,h) + \rho(h,g)$$

Lemma: Assume that $\{f_n\}$ is a sequence in $B(X,Y)$. Then $\{f_n\}$ converges to $f \in B(X,Y)$ if and only if f_n converges uniformly to f.

Proof.

$$f_n \to f \text{ in } B(X,Y)$$
$$\Updownarrow$$
$$\rho(f_n,f) \to 0 \text{ as } n \to \infty.$$
$$\Updownarrow$$
$$\sup\{d_Y(f_n(x), f(x)) : x \in X\} \to 0$$
$$\Updownarrow$$
$$f_n \to f \text{ uniformly.}$$

Theorem: If Y is complete, then $B(X,Y)$ is also complete.

Proof: Let $\{f_n\}$ be a Cauchy sequence in $B(X,Y)$. We need to prove that $f_n \to f$ for some $f \in B(X,Y)$. For $x \in X$,

$$d_Y(f_n(x), f_m(x)) \leq \rho(f_n, f_m)$$

and since $\{f_n\}$ is a Cauchy sequence, so is $\{f_n(x)\}$. Since Y is complete, the sequence $\{f_n(x)\}$ converges to some point $f(x)$ in Y. This defines a function $f: X \to Y$. We need to prove that $f \in B(X,Y)$ and that $f_n \to f$ (w.r.t. to ρ).

Given an $\varepsilon > 0$, there is a N that when $n,m \geq N$, then $\rho(f_n, f_m) < \frac{\varepsilon}{2}$. This means that for all $x \in X$, we have

$$d_Y(f_n(x), f_m(x)) < \frac{\varepsilon}{2}$$

Let $m \to \infty$, to obtain

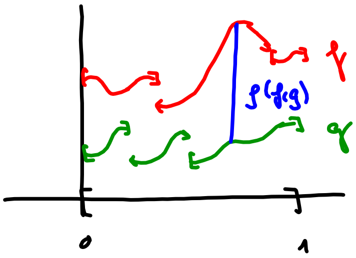
$$d_Y(f_n(x), f(x)) \leq \frac{\varepsilon}{2} \text{ for any } x \in X.$$

Since f_n is bounded, this means that f is bounded and

$$\rho(f_n, f) \leq \frac{\varepsilon}{2} < \varepsilon$$

Hence $\{f_n\}$ converges to f.

Example: $B([0,1], \mathbb{R}) = \{f : [0,1] \to \mathbb{R} : f \text{ is bounded}\}$



$B([0,1], \mathbb{R})$ is complete since $\mathbb{R}$ is complete

Spaces of continuous functions

$C_b(X,Y) = \{f: X \to Y : f \text{ is bounded and continuous}\} \subseteq B(X,Y)$

Proposition: $C_b(X,Y)$ is a closed subset of $B(X,Y)$

Proof: We shall prove that if $\{f_n\}$ is a sequence from $C_b(X,Y)$ converging to f, then $f \in C_b(X,Y)$. Since $f_n \to f$ in ρ, we know $f_n \to f$ uniformly. This implies that f is continuous, and hence $f \in C_b(X,Y)$.

Theorem: If Y is complete, then $C_b(X,Y)$ is complete.

Proof: Since $B(X,Y)$ is complete and $C_b(X,Y)$ is a closed subset, then $C_b(X,Y)$ is complete (by some result in Ch 3).

Proposition: If X is compact, then all continuous functions $f: X \to Y$ are bounded.

Proof: Pick an $a \in X$, we need to prove that there is an $M \in \mathbb{R}$ such that $d_Y(f(a), f(x)) \leq M$ for all $x \in X$. Let us define $g: X \to \mathbb{R}$ by

$$g(x) = d_Y(f(a), f(x)).$$

It clearly suffices to show that g is bounded, and this will follow from the Extremal Value Theorem if we can prove that g is continuous.

$$|g(x) - g(y)| = |d_Y(f(a), f(x)) - d_Y(f(a), f(y))| \leq d_Y(f(x), f(y))$$

by the inverse triangle inequality. Since f is continuous, it follows that g is continuous as any δ that will work for f will also work for g.

Theorem: Assume that X is a compact metric space and let $C(X,Y)$ be the set of all continuous functions $f: X \to Y$. Then

$$\rho(f,g) = \sup\{d_Y(f(x), g(x)) : x \in X\}$$

is a metric on $C(X,Y)$ and if Y is complete, so is $C(X,Y)$.

Proof: Since X is compact, $C(X,Y)$ is the same as $C_b(X,Y)$ and we have proved all these properties for $C_b(X,Y)$

Applications to differential equations

$$y'(t) = f(t, y(t)), \quad y(0) = \underline{y_0}$$

Integrate:

$$\int_0^x y'(t)\,dt = \int_0^x f(t, y(t))\,dt$$

$$\overset{\shortparallel}{y(x) - \underset{\underset{y_0}{\shortparallel}}{y(0)}} = \int_0^x f(t, y(t))\,dt$$

$$\underbrace{y(x) = y_0 + \int_0^x f(t, y(t))\,dt}$$

Define a function $\underline{F} : \underline{C([0,1], \mathbb{R})} \to \underline{C([0,1], \mathbb{R})}$ by

$$z \in C([0,1],\mathbb{R}), \quad F(z) \in C([0,1],\mathbb{R})$$

$$\underline{F(z)(x)} = y_0 + \int_0^x \underline{f(t, z(t))}\,dt$$

compact space

contraction?

If y solves the differential equation:

$$F(y)(x) = y(x) = y_0 + \int_0^x f(t, y(t))\,dt$$

Hence $F(y) = y$. We are looking for fixed points, and Banach Fixed Point Theorem guarantees that fixed points for contractions on complete space