Absolute convergence

Calculus: $\sum_{n=0}^{\infty} a_n$ converges absolutely if $\sum_{n=0}^{\infty} |a_n|$ converges.

Normed spaces: $\sum_{n=0}^{\infty} \vec{v}_n$ converges absolutely if $\sum_{n=0}^{\infty} \|\vec{v}_n\|$

Theorem: The following are equivalent for a normed space V.

(i) V is complete

(ii) All absolutely convergent sequences in V converge.

Proof: (i) $\Rightarrow$ (ii) Assume that V is complete and that $\sum_{n=0}^{\infty} \vec{v}_n$ is absolutely convergent. Since V is complete, it suffices to show that $\sum \vec{v}_n$ is a Cauchy sequence. Let

$$\vec{s}_N = \sum_{n=0}^{N} \vec{v}_n \quad \text{and} \quad S_N = \sum_{n=0}^{N} \|v_n\|$$

Assume that $M > N$:

$$\|\vec{s}_M - \vec{s}_N\| = \left\|\sum_{n=N+1}^{M} \vec{v}_n\right\| \le \sum_{n=N+1}^{M} \|\vec{v}_n\| \le |S_M - S_N|$$

Since $\sum \|\vec{v}_n\|$ converges, it is a Cauchy sequence, and hence we can get the differences $|S_M - S_N|$ as small as we want. Thus we can get $\|\vec{s}_M - \vec{s}_N\|$ as small as we want, which means that $\{\vec{s}_N\}$ is a Cauchy sequence and hence $\sum_{n=0}^{\infty} \vec{v}_n$ converges.

(ii) $\Rightarrow$ (i) Assume that all absolutely convergent ~~sequences~~ series converge and let $\{\vec{v}_n\}$ be a Cauchy sequence. We need to prove that $\{\vec{v}_n\}$ convergent. Since $\{\vec{v}_n\}$ is a Cauchy sequence, there is for every k an N_k such that if $n, m \ge N_k$, then

$$\|\vec{v}_n - \vec{v}_m\| \le \frac{1}{2^k}.$$

We may assume that the sequence $\{N_k\}$ is increasing. In particular, we have

$$\|\vec{v}_{N_{k+1}} - \vec{v}_{N_k}\| \le \frac{1}{2^k}$$

Since $\sum \frac{1}{2^k}$ converges, the series $\sum \|\vec{v}_{N_{k+1}} - \vec{v}_{N_k}\|$ converges, and hence $\sum_{k=0}^{\infty} (\vec{v}_{N_{k+1}} - \vec{v}_{N_k})$ converges absolutely. By assumption, the series $\sum_{k=0}^{\infty} (\vec{v}_{N_{k+1}} - \vec{v}_{N_k})$ converges. The partial sums look like

$$\sum_{k=0}^{K-1} (\vec{v}_{N_{k+1}} - \vec{v}_{N_k}) = (\cancel{\vec{v}_{N_1}} - \vec{v}_{N_0}) + (\cancel{\vec{v}_{N_2}} - \cancel{v_{N_1}}) + (\vec{v}_{N_3} - \cancel{\vec{v}_{N_2}}) + \cancel{\cdots} + (\vec{v}_{N_K} - \cancel{\vec{v}_{N_{K-1}}}) = \vec{v}_{N_K} - \vec{v}_{N_0} \longrightarrow \vec{s}$$

hence $\vec{v}_{N_k} \to \vec{s} - \vec{v}_{N_0}$. This means that a subsequence $\{\vec{v}_{n_k}\}$ of the original sequence converges, and since $\{\vec{v}_n\}$ is a Cauchy sequence, it must also convergence.

Definition: A complete normed space is called a Banach space.

Inner product spaces (Sect. 5.3)

Def: Let V be a linear space over $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$). An inner product on V is a function $\langle\cdot,\cdot\rangle : V\times V \to \mathbb{K}$ such that

(i) $\langle \vec{u},\vec{v}\rangle = \overline{\langle \vec{v},\vec{u}\rangle}$ (if $\mathbb{K}=\mathbb{R}$, we get $\langle \vec{u},\vec{v}\rangle = \langle \vec{v},\vec{u}\rangle$)

(ii) $\langle \vec{u}+\vec{v},\vec{w}\rangle = \langle \vec{u},\vec{w}\rangle + \langle \vec{v},\vec{w}\rangle$

(iii) $\langle \alpha\vec{u},\vec{v}\rangle = \alpha\langle \vec{u},\vec{v}\rangle$

(iv) $\langle \vec{u},\vec{u}\rangle \geq 0$ with equality only if $\vec{u}=\vec{0}$.

$\langle u,u\rangle \overset{(i)}{=} \overline{\langle u,u\rangle}$

hence $\langle u,u\rangle \in \mathbb{R}$

Some consequences

(v) $\langle \vec{u},\vec{v}+\vec{w}\rangle = \langle \vec{u},\vec{v}\rangle + \langle \vec{u},\vec{w}\rangle$

(vi) $\langle \vec{u},\alpha\vec{v}\rangle = \bar{\alpha}\langle \vec{u},\vec{v}\rangle$

(vii) $\langle \alpha\vec{u},\alpha\vec{v}\rangle = |\alpha|^2\langle \vec{u},\vec{v}\rangle$

Proof (vi): $\langle \vec{u},\alpha\vec{v}\rangle \overset{(i)}{=} \overline{\langle \alpha\vec{v},\vec{u}\rangle} \overset{(iii)}{=} \overline{\alpha\langle \vec{v},\vec{u}\rangle} = \bar{\alpha}\,\overline{\langle \vec{v},\vec{u}\rangle}$
$\overset{(i)}{=} \bar{\alpha}\langle \vec{u},\vec{v}\rangle$.

Examples: $\mathbb{R}^n$: $\langle \vec{v},\vec{w}\rangle = \vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2 + \dots + v_nw_n$

$\mathbb{C}^n$: $\langle \vec{v},\vec{w}\rangle = v_1\overline{w_1} + v_2\overline{w_2} + \dots + v_n\overline{w_n}$

$C([a,b],\mathbb{R})$: $\langle f,g\rangle = \int_a^b f(t)g(t)\,dt$

$C([a,b],\mathbb{C})$: $\langle f,g\rangle = \int_a^b f(t)\overline{g(t)}\,dt$

Def: The norm generated by an innerproduct is defined by
$\|\vec{u}\| = \langle \vec{u},\vec{u}\rangle^{1/2}$ (is this really a norm?).

Def: Two elements $\vec u,\vec v\in V$ are orthogonal if $\langle \vec u,\vec v\rangle=0$.

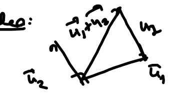
Pythagorean Theorem: Assume that $\vec v_1,\vec v_2,\dots,\vec v_n$ are orthogonal. Then

$$\|\vec v_1+\vec v_2+\dots+\vec v_n\|^2=\|\vec v_1\|^2+\|\vec v_2\|^2+\dots+\|\vec v_n\|^2$$

Proof. $\|\vec v_1+\vec v_2+\dots+\vec v_n\|^2=\langle \vec v_1+\dots+\vec v_n,\vec v_1+\dots+\vec v_n\rangle$

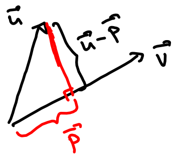
$$=\sum_{i,j}\langle\vec v_i,\vec v_j\rangle=\sum_{i=1}^n\langle \vec v_i,\vec v_i\rangle=\sum_{i=1}^n\|\vec v_i\|^2$$

Idea:



$$\|\vec u_1+\vec u_2\|^2=\|\vec u_1\|^2+\|u_2\|^2$$

Projection:



Projection of $\vec u$ on $\vec v$

(i) $\vec p$ parallel to $\vec v$, $\vec p=\alpha\vec v$

(ii) $\vec u-\vec p$ orthogonal to $\vec v$

$$0=\langle\vec u-\vec p,\vec v\rangle=\langle\vec u-\alpha\vec v,\vec v\rangle=\langle\vec u,\vec v\rangle-\alpha\langle\vec v,\vec v\rangle$$

Solving for α: $\alpha=\dfrac{\langle\vec u,\vec v\rangle}{\langle\vec v,\vec v\rangle}=\dfrac{\langle\vec u,\vec v\rangle}{\|\vec v\|^2}$

Proposition: The projection of $\vec u$ on $\vec v$ is given by

$$\vec p=\frac{\langle\vec u,\vec v\rangle}{\|\vec v\|^2}\vec v.$$

The length of $\vec p$ is

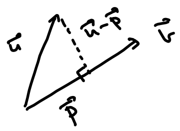
$$\|\vec p\|=\left\|\frac{\langle\vec u,\vec v\rangle}{\|v\|^2}\vec v\right\|=\frac{|\langle\vec u,\vec v\rangle|}{\|\vec v\|^{\cancel{2}}}\cancel{\|\vec v\|}=\frac{|\langle\vec u,\vec v\rangle|}{\|\vec v\|}.$$

Cauchy-Schwarz inequality:

$$|\langle\vec u,\vec v\rangle|\le\|\vec u\|\|\vec v\|$$

with equality if and only the vectors $\vec u,\vec v$ are parallel or one of them is zero.

Proof: By the Pythagorean Theorem



$$\|\vec u\|^2=\|\vec p\|^2+\|\vec u-\vec p\|^2\ge\|\vec p\|^2=\frac{|\langle\vec u,\vec v\rangle|^2}{\|\vec v\|^2}\quad\Big|\ \|\vec v\|^2$$

Hence

$$\|\vec u\|^2\|\vec v\|^2\ge|\langle\vec u,\vec v\rangle|^2$$

which implies $\|\vec u\|\|\vec v\|\ge|\langle\vec u,\vec v\rangle|$.

Triangle inequality: If $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ is the associated norm, then

$$\|\vec u+\vec v\|\le\|\vec u\|+\|\vec v\|$$

Proof. $\|\vec u+\vec v\|^2=\langle\vec u+\vec v,\vec u+\vec v\rangle=\overbrace{\langle\vec u,\vec u\rangle}^{\|\vec u\|^2}+\overbrace{\langle\vec u,\vec v\rangle}^{\le\|\vec u\|\|\vec v\|}+\overbrace{\langle\vec v,\vec u\rangle}^{\le\|\vec u\|\|\vec v\|}+\overbrace{\langle\vec v,\vec v\rangle}^{\|\vec v\|^2}$

$$\le\|\vec u\|^2+2\|\vec u\|\|\vec v\|+\|\vec v\|^2=(\|\vec u\|+\|\vec v\|)^2$$

hence

$$\|\vec u+\vec v\|\le\|\vec u\|+\|\vec v\|$$

Theorem: The norm $\|\cdot\|$ derived from an inner product really is a norm!

Proof: We need to check the three conditions.

(i) $\|\vec u\|\ge0$ with equality if and only if $\vec u=\vec 0$

Check: $\|\vec u\|=\langle\vec u,\vec u\rangle^{1/2}\ge0$ and only 0 when $\vec u=\vec 0$

(ii) $\|\alpha\vec u\|=|\alpha|\|\vec u\|$

Check: $\|\alpha\vec u\|=\langle\alpha\vec u,\alpha\vec u\rangle^{1/2}=(\alpha\bar\alpha\langle\vec u,\vec u\rangle)^{1/2}=(|\alpha|^2\langle\vec u,\vec u\rangle)^{1/2}$

$=|\alpha|(\langle\vec u,\vec u\rangle)^{1/2}=|\alpha|\|\vec u\|$

$z=x+iy$

$\bar z=x-iy$

$z\bar z=\dots$

(iii) $\|\vec u+\vec v\|\le\|\vec u\|+\|\vec v\|$

Proposition: Assume that $\|\cdot\|$ is the norm generated by $\langle \cdot,\cdot \rangle$.

(i) If $\vec{u}_n \to \vec{u}$, then $\|\vec{u}_n\| \to \|\vec{u}\|$

(ii) If $\sum_{n=0}^{\infty} \vec{u}_n$ converges, then $\|\sum_{n=0}^{\infty} \vec{u}_n\| = \lim_{N\to\infty} \|\sum_{n=0}^{N} \vec{u}_n\|$

(iii) If $u_n \to u$, then $\langle u_n, v\rangle \to \langle u, v\rangle$

(iv) $\langle \sum_{n=0}^{\infty} \vec{u}_n, v\rangle = \sum_{n=0}^{\infty} \langle \vec{u}_n, \vec{v}\rangle$

Proof: (i) Proved for normed spaces (it says $\lim \|\vec{u}_n\| = \|\overbrace{\lim \vec{u}_n}^{\vec{u}}\|$

(ii) $\lim \|\sum_{n=0}^{N} \vec{u}_n\| \overset{(i)}{=} \|\lim_{N\to\infty} \sum_{n=0}^{N} \vec{u}_n\| = \|\sum_{n=0}^{\infty} \vec{u}_n\|$

(iii) $\underbrace{|\langle \vec{u}_n, \vec{v}\rangle - \langle \vec{u}, \vec{v}\rangle|}_{\downarrow\, 0} = |\langle \vec{u}_n - \vec{u}, \vec{v}\rangle| \overset{CS}{\leq} \underbrace{\|\vec{u}_n - \vec{u}\|}_{\downarrow\, 0} \|\vec{v}\| \to 0$

(it says: $\lim_{n\to\infty} \langle \vec{u}_n, \vec{v}\rangle = \langle \overbrace{\lim_{n\to\infty} \vec{u}_n}^{\vec{u}}, \vec{v}\rangle$

(iv) $\langle \sum_{n=0}^{\infty} \vec{u}_n, \vec{v}\rangle = \langle \lim_{N\to\infty} \sum_{n=0}^{N} \vec{u}_n, \vec{v}\rangle = \lim_{N\to\infty} \langle \sum_{n=0}^{N} \vec{u}_n, \vec{v}\rangle$

$= \lim_{N\to\infty} \sum_{n=0}^{N} \langle \vec{u}_n, \vec{v}\rangle = \sum_{n=0}^{\infty} \langle \vec{u}_n, \vec{v}\rangle$