7$^{\text{th}}$ February, 2022

MAT2400

Mandatory assignment 1 of 2

Submission deadline

Thursday 24$^{\text{th}}$ February 2022, 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course, and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. You only have one attempt at each assignment, and you need to have both assignments approved in order to take the exam. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

There is a new regime for the mandatory assignments this year. In the new regime you only have one attempt at each assignment and not two as in earlier years. As the purpose of the new regime is to handle the assignments in a more efficient and pedagogical way and not to fail more students, we shall put more emphasis on effort *in the grading this year: As long as you have documented that you have made a serious attempt at the majority of the problems, we will pass you. The best way to document that you have tried, is, of course, to solve the problems, but you can also do it by telling us what you have tried and why it failed. We encourage you to discuss, collaborate, and help each other. Do not hesitate to contact the teachers (preferably well in advance of the deadline) if you have problems.*

Problem 1. Let X be the set of all sequences $\{x_n\}_{n\in\mathbb{N}}$ of real numbers such that $\lim_{n\to\infty} x_n = 0$.

a) Use the definition of convergence to show that if $\{x_n\} \in X$, then there is a $K \in \mathbb{N}$ such that $|x_K| = \sup\{|x_n| \colon n \in \mathbb{N}\}$ (i.e. x_K is an element of maximal absolute value).

b) Define $d\colon X \times X \to [0, \infty)$ by

$$d(\{x_n\}, \{y_n\}) = \sup\{|x_n - y_n| \colon n \in \mathbb{N}\}.$$

Show that d is a metric on X.

c) Let Y be the set of all sequences $\{y_n\}_{n\in\mathbb{N}}$ of real numbers such that $\sum_{n=1}^{\infty} |y_n| < \infty$. Show that $Y \subseteq X$. Find a sequence $\{x_n\}$ that belongs to X but not to Y (you can use everything you know from calculus).

d) Assume $\{x_n\} \in X \setminus Y$ and let $\epsilon > 0$. Show that the ball $B(\{x_n\}; \epsilon)$ contains elements from Y. Explain why this shows that Y is not closed.

e) Assume $\{y_n\} \in Y$ and let $\epsilon > 0$. Show that $B(\{y_n\}; \epsilon)$ contains elements from $X \setminus Y$. Explain why this shows that Y is not open.

Problem 2. A metric space (X, d) is called *disconnected* if there are two non-empty, open subsets O_1, O_2 such that $O_1 \cup O_2 = X$ and $O_1 \cap O_2 = \emptyset$.

a) Let $X = [0, 1] \cup [2, 3]$ have the usual metric $d(x, y) = |x - y|$. Show that (X, d) is disconnected.

b) Show that $\mathbb{Q}$ with the usual metric $d(x, y) = |x - y|$ is disconnected. (Hint: Consider $O_1 = \{x \in \mathbb{Q} \colon x^2 > 2\}$ and $O_2 = \{x \in \mathbb{Q} \colon x^2 < 2\}$.)

c) Assume that (X, d) is a connected (i.e. not disconnected) metric space and that $f \colon X \to \mathbb{R}$ is a continuous function such that there are two points $a, b \in X$ with $f(a) < 0 < f(b)$. Show that there is point $c \in X$ such that $f(c) = 0$. (This is an abstract version of the Intermediate Value Theorem.)

GOOD LUCK!