MAT2400: Solution to Mandatory Assignment 2, Spring 2022

After some of the problems I have added comments in blue to discuss common mistakes or alternative approaches.

Problem 1.

a) According to section 10.1, the real Fourier series of f is of the form

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

where

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos nx\,dx$$
$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}|x|\sin nx\,dx$$

We first observe that since $f(x)=|x|$ is an even function, the integrands $|x|\sin nx$ are odd, and hence all the b_n's are zero. Turning to the a_n's, we observe that since $\cos nx$ is even, so is $|x|\cos nx$, and hence by symmetry

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx$$

We first compute a_0:

$$a_0=\frac{2}{\pi}\int_0^{\pi}x\cos 0\,dx=\frac{2}{\pi}\int_0^{\pi}x\,dx=\frac{2}{\pi}\cdot\frac{\pi^2}{2}=\pi$$

For $n>0$, we get

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx$$

If we use integration by parts with $u=x$ and $v'=\cos nx$, we get $u'=1$ and $v=\frac{\sin nx}{n}$, and hence

$$a_n=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx=\frac{2}{\pi}\left(\left[x\cdot\frac{\sin nx}{n}\right]_0^{\pi}-\int_0^{\pi}1\cdot\frac{\sin nx}{n}\,dx\right)$$
$$=\frac{2}{\pi}\left(0-\left[-\frac{\cos nx}{n^2}\right]_0^{\pi}\right)=\frac{2}{\pi n^2}(\cos(n\pi)-1)$$

Observing that $\cos n\pi$ equals -1 when n is odd and 1 when n is even, we get

$$a_n=\begin{cases}0 & \text{for } n \text{ even}\\ -\frac{4}{\pi n^2} & \text{for } n \text{ odd}\end{cases}$$

Using that an odd number is of the form $2n+1$ for an integer n, we get that the real Fourier series of f is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos[(2n+1)x]}{(2n+1)^2}.$$

As this problem illustrates, the computation of Fourier series can often be simplified by exploiting the symmetry of the functions involved. Some of you have solved the problem by first finding the complex Fourier series, but that is hardly more efficient here.

b) If we put $x = 0$ in the formula

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos[(2n+1)x]}{(2n+1)^2},$$

we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Rearranging the terms, we see that

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n+1)^2} + \cdots$$

It's also possible to use $x = \pm\pi$.

c) The figure below shows the graphs of f and the three approximations. The function f is in black, the approximations for $N = 0$, $N = 1$, and $N = 2$ in blue, green, and red, respectively. Note that the bigger N gets, the closer the approximation follows the graph, especially at the "corners".

Problem 2. a) We have

$$\begin{aligned} \|\mathbf{e}_n - \mathbf{e}_m\|^2 &= \langle \mathbf{e}_n - \mathbf{e}_m, \mathbf{e}_n - \mathbf{e}_m \rangle \\ &= \langle \mathbf{e}_n, \mathbf{e}_n \rangle - \langle \mathbf{e}_n, \mathbf{e}_m \rangle - \langle \mathbf{e}_m, \mathbf{e}_n \rangle + \langle \mathbf{e}_m, \mathbf{e}_m \rangle \\ &= 1 + 0 + 0 + 1 = 2 \end{aligned}$$

and hence $\|\mathbf{e}_n - \mathbf{e}_m\| = \sqrt{2}$.

It is also possible to use the Pythagorean Theorem here.

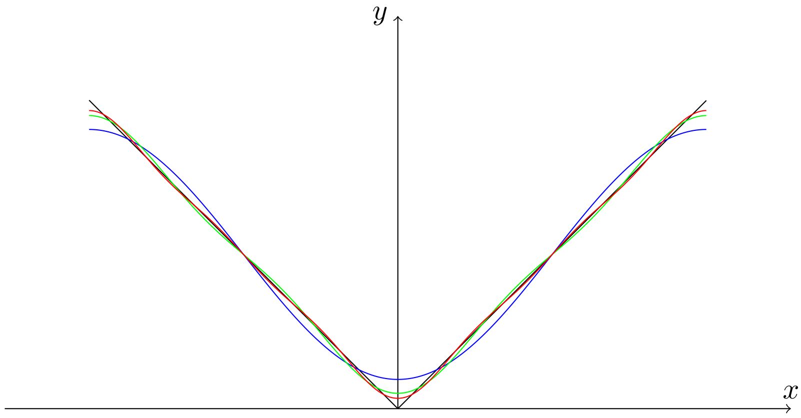

Figure 1: Function f in black, approximations $N = 0$, $N = 1$, $N = 2$ in blue, green, and red, respectively.

b) Note that $\{\mathbf{e}_n\}$ is a sequence in S. If $\{\mathbf{e}_{n_k}\}$ is any subsequence of $\{\mathbf{e}_n\}$, part a) tells us that if $k \neq l$, then $\|\mathbf{e}_{n_k} - \mathbf{e}_{n_l}\| = \sqrt{2}$. This means that the subsequence $\{\mathbf{e}_{n_k}\}$ is not a Cauchy sequence and hence cannot converge. Thus $\{\mathbf{e}_n\}$ is a sequence in S without any convergent subsequence, which means that S isn't compact.

It is also possible to solve this problem by using the open covering property, but that seems a little less natural to me.

Problem 3. a) By definition, $u_0(x) = v_0(x) = 1$. To normalize, we compute the norm: $\|u_0\| = \left(\int_0^1 1^2\,dx\right)^{1/2} = 1$. Hence $e_0(x) = u_0(x) = 1$. The next step is to compute u_1:

$$u_1(x) = v_1(x) - \frac{\langle v_1, u_0\rangle}{\|u_0\|^2} u_0(x) = x - \langle v_1, u_0\rangle = x - \int_0^1 x \cdot 1\,dx = x - \frac{1}{2}$$

To normalize, we must compute the norm of u_1:

$$\|u_1\| = \left(\int_0^1 \left(x - \frac{1}{2}\right)^2 dx\right)^{1/2} = \left(\left[\frac{1}{3}\left(x - \frac{1}{2}\right)^3\right]_0^1\right)^{1/2} = \frac{1}{2\sqrt{3}}.$$

Hence $e_1(x) = 2\sqrt{3}(x - \frac{1}{2})$.

b) Let V_n be the space of all polynomials of degree n or less. As any element in V_n can be written uniquely as a sum $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, the original polynomials $v_0(x) = 1, v_1(x) = x, \ldots, v_n(x) = x^n$ form a basis for V_n. Since $\mathrm{Span}(e_0, e_1, \ldots e_n) = \mathrm{Span}(v_0, v_1, \ldots, v_n)$, the new set $e_0(x), e_1(x), e_2(x), \ldots, e_n(x)$ is also a basis for V_n. Hence any polynomial

$p \in V_n$ can be written as a sum $p(x) = c_n e_n(x) + c_{n-1}e_{n-1}(x) + \cdots + c_0$. Thus if $\langle h, e_i \rangle = 0$ for all i, we also have

$$\langle h, p \rangle = \langle h, c_n e_n + c_{n-1}e_{n-1} + \cdots + c_0 e_0 \rangle$$

$$= c_n \langle h, e_n \rangle + c_{n-1}\langle h, e_{n-1} \rangle + \cdots + c_0 \langle h, e_0 \rangle = 0.$$

Many got the basis argument wrong here. It isn't sufficient to say that something is a basis; you have to say what it is a basis of – and prove it! Also, many of those who were a bit careless with the arguments, ended up stating that any element h in V can be written as a sum $\sum_{n=0}^{\infty} a_n e_n$ for a sequence of numbers $\{a_n\}$. This statement is false or true depending on how one interprets it. If it means that the series $\sum_{n=0}^{\infty} a_n e_n(x)$ converges pointwise to $h(x)$ for all x, it is false as there are many continuous functions h that are not the sum of a power series (sums of power series are differentiable, and there are many continuous functions that aren't differentiable). If, on the other hand, it means that every h in V is the sum of a series $\sum_{n=0}^{\infty} a_n e_n$ with respect to the norm $\| \cdot \|_2$ coming from the inner product (i.e. $\lim_{N\to\infty} \|h - \sum_{n=0}^{N} a_n e_n\|_2 = 0$), then the statement is actually true, but needs to be proved (it's equivalent to proving that $\{e_n\}$ is an orthonormal basis for V). This shows how treacherous the notation $\sum_{n=0}^{\infty} a_n e_n$ can be – the sum depends on a sense of convergence that doesn't show in the notation.

I'll give two solutions of 3c) and 3d), the first based primarily on Weierstrass's Approximation Theorem, and the second on abstract Fourier analysis.

Solution I

c) By the Weierstrass Approximation Theorem, there is a sequence $\{p_n\}$ of polynomials converging uniformly to h. By the Extremal Value Theorem there is a number M such that $|h(x)| \le M$ for all $x \in [0, 1]$. Hence

$$\left|\int_0^1 h(x)^2\, dx - \int_0^1 h(x)p_n(x)\, dx\right| = \left|\int_0^1 h(x)(h(x) - p_n(x))\, dx\right|$$

$$\le \int_0^1 |h(x)||h(x) - p_n(x)|\, dx \le M\rho(h, p_n)$$

where $\rho(h, p_n) = \sup\{|h(x) - p_n(x)| \colon x \in [0, 1]\}$ is the usual supremum metric. As $\rho(h, p_n) \to 0$, we see that $\int_0^1 h(x)^2\, dx = \lim_{n\to\infty} \int_0^1 h(x)p_n(x)\, dx$, and since $\int_0^1 h(x)p_n(x)\, dx = 0$ for all n, we must have $\langle h, h \rangle = \int_0^1 h(x)^2\, dx = 0$. Since V is an inner product space, it follows that $h = 0$.

Many solved this problem by writing:

$$\int_0^1 h(x)^2\,dx = \int_0^1 h(x) \lim_{n\to\infty} p_n(x)\,dx = \lim \int_0^1 f(x)p_n(x)\,dx = 0,$$

but then one has to justify that one can pull the limit outside the integral (recall from section 4.3 that this is not always the case). A possible attempt is to rewrite the derivation in terms of inner products

$$\int_0^1 h(x)^2\,dx = \langle h, h\rangle = \langle h, \lim_{n\to\infty} p_n\rangle = \lim_{n\to\infty}\langle h, p_n\rangle = 0$$

and then refer to Proposition 5.7.3(iii) to justify pulling out the limit. The problem with this is that the limit in 5.3.7 is a limit in terms of the norm coming from the inner product, while our limit is in terms of uniform convergence. In order to use 5.3.7, one first has to prove that uniform convergence implies convergence in norm (which is basically what I did above).

d) First observe that

$$\langle g, e_n\rangle = \langle \sum_{i=0}^{\infty} \alpha_i e_i, e_n\rangle = \sum_{i=0}^{\infty}\langle \alpha_i e_i, e_n\rangle = \alpha_n$$

where we have used Proposition 5.3.7(iv) to pull the sum outside. If we now put $h = f - g$, then h is continuous since f and g are, and we can apply b) and c) to h: As

$$\langle h, e_n\rangle = \langle f, e_n\rangle - \langle g, e_n\rangle = \alpha_n - \alpha_n = 0$$

for all n, we get that $h = 0$ and hence $f = g$.

This solution turned out to be surprisingly hard to find. I thought it would be just putting the ball in the open net ...

Solution II

The idea of this solution is to use that $\{e_0, e_0, e_2 \ldots, e_n, \ldots\}$ is a basis for V, but this isn't something one can just take for granted, it's something one has to prove. So let us prove this first:

Theorem: $\{e_0, e_0, e_2 \ldots, e_n, \ldots\}$ is an orthonormal basis for V.

Proof: According to the definition (textbook page 148), we have to show that each element $f \in V$ can be written as a sum $f = \sum_{n=0}^{\infty} \alpha_n e_n$ in a unique way. The uniqueness is easy: If $f = \sum_{n=0}^{\infty} \alpha_n e_n = \sum_{n=0}^{\infty} \beta_n e_n$, we get

$$0 = \|f - f\|^2 = \|\sum_{n=0}^{\infty} (\alpha_n - \beta_n) e_n\|^2 = \sum_{n=0}^{\infty} (\alpha_n - \beta_n)^2$$

which implies that $\alpha_n = \beta_n$ for all n.

If we can prove that $f = \sum_{n=0}^{\infty} \alpha_n e_n$, where $\alpha_n = \langle f, e_n \rangle$, we will be done. This means that given an $\epsilon > 0$, we must show that there is an $N \in \mathbb{N}$ such that $\|f - \sum_{n=0}^{k} \alpha_n e_n\|_2 < \epsilon$ for all $k \geq N$ (here $\|\cdot\|_2$ denotes the norm of the inner product). By Weierstrass's Approximation Theorem 4.10.1, there is a polynomial q such that $|f(x) - q(x)| < \epsilon$ for all $x \in [0, 1]$. But then

$$\|f - q\|_2 = \left(\int_0^1 |f(x) - q(x)|\, dx \right)^{\frac{1}{2}} < \left(\int_0^1 \epsilon\, dx \right)^{\frac{1}{2}} = \epsilon$$

Let N be the degree of q. If $k \geq N$, q lies in $\mathrm{Span}\{e_0, e_1, \ldots, e_k\}$, and according to Proposition 5.3.8, we have

$$\|f - \sum_{n=0}^{k} \alpha_n e_n\|_2 \leq \|f - q\|_2 < \epsilon$$

which is what we needed to prove. (Note that this argument is almost identical to the proof of Corollary 10.2.3.)

It is now easy to prove c) and d):

c) We have just proved that since $h \in V$, $h = \lim_{k \to \infty} \sum_{n=0}^{k} \beta_n e_n$ where $\beta_n = \langle h, e_n \rangle$, and the convergence is in the norm of the inner product. Hence

$$\langle f, f \rangle = \langle f, \lim_{k \to \infty} \sum_{n=0}^{k} \beta_n e_n \rangle = \lim_{k \to \infty} \langle f, \sum_{n=0}^{k} \beta_n e_n \rangle = 0$$

where we have used Proposition 5.3.7(iii) to pull the limit outside the inner product and c) to get the final equality.

d) This part follows immediately from the theorem: As we have both $f = \sum_{n=0}^{\infty} \alpha_n e_n$ (this is what we actually proved in the theorem) and $g = \sum_{n=0}^{\infty} \alpha_n e_n$, we clearly have $f = g$.

Many solved d) by using Parseval's Theorem:

$$\|f - g\|_2^2 = \|\sum_{n=0}^{\infty} (\alpha_n - \alpha_n) e_n\|_2^2 = \sum_{n=0}^{\infty} 0^2 = 0,$$

but Parseval's Theorem requires that $\{e_n\}$ is a basis (otherwise we only have Bessel's Inequality which gives the useless result $\|f - g\|_2 \geq 0$).

But why is it so important that $\{e_0, e_0, e_2 \ldots, e_n, \ldots\}$ is a basis? Well, if $\{e_0, e_1, e_2 \ldots, e_n, \ldots\}$ hadn't been a basis, there could have been an element e in V orthogonal to all the e_n. If we had put $f = g + e$, we would then have

$$\langle f, e_n \rangle = \langle g + e, e_n \rangle = \langle g, e_n \rangle + \langle e, e_n \rangle = \langle g, e_n \rangle,$$

and hence f and g would have had the same Fourier coefficients with respect to $\{e_n\}$ even though they are not equal.