Answers to problems in exam MAT2410 2012

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Problem 1. Find all solutions of the equation $4z^2 + 4z + 5 = 0$.
Solution. We may use the formula for quadratic equations and find

$$z = \frac{-4 \pm \sqrt{16 - 80}}{8} = -\frac{1}{2} \pm i\,.$$

Problem 2. Find the Laurent series for

$$f(z) = \frac{1}{z(4 - z^2)}$$

in the domains $0 < |z| < 2$ and $|z| > 2$.
Solution. If $|z| < 2$ then

$$\frac{1}{4 - z^2} = \frac{1}{4} \cdot \frac{1}{1 - \frac{z^2}{4}}$$

and $\frac{|z|^2}{4} < 1$. Thus

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{4^{n+1}} z^{2n-1} = \frac{1}{4} z^{-1} + \frac{1}{16} z + \frac{1}{64} z^3 + \cdots\,.$$

If $|z| > 2$ then

$$\frac{1}{4 - z^2} = \frac{-1}{z^2} \cdot \frac{1}{1 - \frac{4}{z^2}}$$

and $\frac{4}{|z|^2} < 1$. Thus

$$f(z) = -\sum_{n=0}^{\infty} 4^n z^{-2n-3} = -z^{-3} - 4z^{-5} - 16z^{-7} + \cdots\,.$$

Problem 3. Let

$$f(z) = \frac{z^2 + 1}{z^3 + i}\,.$$

a) Find all the singularities of $f(z)$ and determine their type.

b) Compute the residues of $f(z)$ at all of the singularities.

Solution.
a) Note first that i is a zero of both z^2+1 and z^3+i so it is a removable singularity. Polynomial division yields

$$z^3 + i = (z-i)(z^2 + iz - 1)\,.$$

Either by solving $z^3 = e^{i3\pi/2}$ or using the formula for $z^2 + iz - 1 = 0$ one finds the two singularities

$$e^{i\frac{7\pi}{6}} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i \quad \text{and} \quad e^{i\frac{11\pi}{6}} = \frac{\sqrt{3}}{2} - \frac{1}{2}i\,.$$

They are both simple poles.
b) Since the poles are simple we may use the formula

$$\operatorname{Res}\left(\frac{p(z)}{q(z)}; z_0\right) = \frac{p(z_0)}{q'(z_0)}$$

and compute

$$\operatorname{Res}(f(z); -\frac{\sqrt{3}}{2} - \frac{1}{2}i) = \frac{1}{2} - \frac{\sqrt{3}}{6}i$$

and

$$\operatorname{Res}(f(z); \frac{\sqrt{3}}{2} - \frac{1}{2}i) = \frac{1}{2} + \frac{\sqrt{3}}{6}i\,.$$

Problem 4. Let n be a positive integer and set

$$f(z) = \frac{1}{(1+z^2)^n}\,.$$

a) Show that

$$\operatorname{Res}(f; i) = -\frac{2i}{4^n}\binom{2n-2}{n-1}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial number.

b) Compute the real integal

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}\,.$$

Solution.
a) The function $f(z)$ has a pole of order n at i. Using the formula for residues at poles we get

$$\begin{aligned}
\operatorname{Res}(f(z);i) &= \lim_{z\to i} \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\frac{(z-i)^n}{(1+z^2)^n} \\
&= \lim_{z\to i} \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\frac{1}{(z+i)^n} \\
&= \frac{1}{(n-1)!}\frac{(-n)(-n-1)\cdots(-2n+2)}{(2i)^{2n-1}} \\
&= \frac{1}{(n-1)!}\frac{(-1)^{n-1}\,n(n+1)\cdots(2n-2)}{2^{2n-1}(-1)^n(-i)} \\
&= -\frac{2i}{4^n}\frac{n(n+1)\cdots(2n-2)}{(n-1)!} = -\frac{2i}{4^n}\binom{2n-2}{n-1}.
\end{aligned}$$

b) If I is the integral then it satisfies the conditions for using the formula

$$I = 2\pi i \sum_{z_0 \in \text{ upper half-plane}} \operatorname{Res}(f(z);z_0).$$

where $f(z)$ is the complex function $\frac{1}{(z^2+1)^3}$. Thus $I = \operatorname{Res}(f(z);i) = 3\pi/8$ by setting $n=3$ in a).

Problem 5. Let a and b be complex numbers and set $f(z) = z^3 + az + b$.

a) Show that if $|a| + |b| < 1$ then $f(z)$ has three zeroes counted with multiplicity in the open unit disk $|z| < 1$.

b) Let t be a complex number and set $a = -3t^2$ and $b = 2t^3$, i.e. now $f(z) = z^3 - 3t^2z + 2t^3$. Find a positive real number r so that $|t| < r$ implies that $f(z)$ has three zeroes counted with multiplicity in the open unit disk. Express these zeroes in terms of t.

Solution.
a) If $|z| = 1$ and $|a| + |b| < 1$ then $|z|^3 = 1$ and $|az + b| \le |a| + |b| < 1$ so Rouché's theorem yields the result.
b) From a) we know that the number r can be chosen so that $3|t|^2 + 2|t|^3 < 1$ when $|t| < r$. Consider the real function $g(x) = 3x^2 + 2x^3$. One computes that g is increasing when $x \ge 0$ and that $g(1/2) = 1$ so if $r = 1/2$ then f has 3 zeroes in the disk.

We see that $f(t) = 0$, but also $f'(t) = 3t^2 - 3t^2 = 0$ so $z = t$ is a zero of multiplicity 2. We may find the last zero by polynomial division and get

$$z^3 - 3t^2z + 2t^3 = (z-t)^2(z+2t).$$

Thus $z = -2t$ is also a zero in the disk.

Problem 6. Recall that a fixed point for a function $f(z)$ is a point z_0 with $f(z_0) = z_0$. Consider the Möbius transformation

$$f(z) = \frac{z+i}{iz+1}\,.$$

a) Find the fixed points of $f(z)$.

b) Describe the image of the open unit disk under $f(z)$.

Solution.
a) Fixed points of $f(z)$ are the solutions of the quadratic equation $z + i = z(iz + 1)$ which is equivalent to $i = iz^2$. Thus the answer is $z = \pm 1$.
b) The circle $|z| = 1$ maps to a line since $z = i$ is a pole for f and lies on the circle. Since the fixed points 1 and -1 also lie on this circle the image line must pass through 1 and -1. It is therefore the real axis. The point 0 is in the open unit disk and $f(0) = i$ is in the upper half-plane, therefore the whole open unit disk maps to the upper half-plane, i.e. the set of complex numbers where the imaginary part is positive.