CAUCHY'S THEOREM FOR STAR-SHAPED REGIONS

These notes give an alternative treatment and proof of Cauchy's integral formula without requiring the concept of *toy contours*, as used in Stein and Shakarchi.

Definition 1. We say a subset Ω of $\mathbb{C}$ is *star-shaped* with respect to the point $p \in \Omega$ if, for every point $z \in \Omega$, the line segment from p to z lies entirely inside of Ω.

Note that a star-shaped set is connected, since any pair of points $z_1, z_2 \in \Omega$ can be linked by a line segment from z_1 to p followed by a line segment from p to z_2. Hence, every open star-shaped set is actually a region.

Theorem 1. *Let $\Omega \subset \mathbb{C}$ be a star-shaped region and $f : \Omega \to \mathbb{C}$ be a holomorphic function. Then f has a primitive in Ω.*

Proof. Suppose that Ω is star-shaped with respect to $p \in \Omega$. For $z \in \Omega$, let $\gamma_z \subset \Omega$ be the line segment from p to z. Define

$$F(z) = \int_{\gamma_z} f(w)dw\,.$$

Then $F : \Omega \to \mathbb{C}$ is a primitive for f.

To see this, fix $z \in \Omega$ and let D be a disc centred at z, small enough so that $D \subset \Omega$. Let $h \in \mathbb{C}$ be sufficiently small so that $z + h \in D$. Then, by Goursat's theorem,

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)dw - \int_{\gamma_z} f(w)dw = \int_\eta f(w)dw\,,$$

where η is the line segment from z to $z + h$. (Note that η is contained in D and hence also in Ω.)

Writing $f(w) = f(z) + (f(w) - f(z))$, we have

$$\frac{1}{h}(F(z+h) - F(z)) = \frac{1}{h}\int_\eta f(z)dw + \frac{1}{h}\int_\eta (f(w) - f(z))dw\,.$$

Now, since w is a primitive for the constant function 1, we have $\int_\eta f(z)dw = f(z)\int_\eta 1dw = hf(z)$, so that

$$\frac{1}{h}\int_\eta f(z)dw = f(z)\,.$$

We also have

$$\left|\int_\eta (f(w) - f(z))dw\right| \le \sup_{w\in\eta}|f(w) - f(z)| \cdot \text{length}(\eta) = |h| \sup_{w\in\eta}|f(w) - f(z)|\,,$$

and therefore

$$\left|\frac{1}{h}\int_\eta (f(w) - f(z))dw\right| \le \sup_{w\in\eta}|f(w) - f(z)|\,,$$

which goes to 0 as $h \to 0$ (and thus $w \to z$), due to the continuity of f at z.

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We finally obtain

$$\lim_{h\to 0}\frac{1}{h}(F(z+h)-F(z))=f(z)$$

as required. □

Corollary 2 (Cauchy's theorem for star-shaped regions)**.** *Let $f:\Omega\to\mathbb{C}$ be a holomorphic function defined on a star-shaped region Ω. Then*

$$\int_\gamma f=0$$

for every closed curve γ in Ω.

Proof. By the previous result, f has a primitive on Ω. □

Theorem 3 (Cauchy's integral formula for a disc)**.** *Let f be holomorphic in an open set Ω that contains an open disc D and its boundary circle C, equipped with the positive (counterclockwise) orientation. Then*

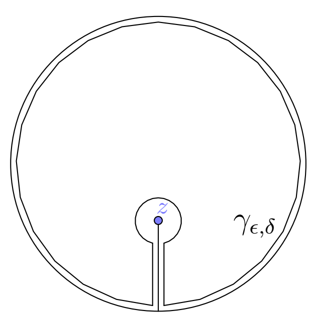
$$f(z)=\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{\zeta-z}d\zeta$$

for every $z\in D$.

Proof. Let $z\in D$ be given, and form the open set $\tilde{D}$ obtained by removing from D the point z, together with the line segment from z radially outwards to the boundary circle C. Then $\tilde{D}$ is a star-shaped region with respect to the centre of D. Define $g(\zeta)=\frac{f(\zeta)}{\zeta-z}$ for $\zeta\in\Omega\setminus\{z\}$. Then $g|_{\tilde{D}}:\tilde{D}\to\mathbb{C}$ is a holomorphic function, so we may apply Cauchy's theorem for star-shaped regions to obtain

$$\int_{\gamma_{\epsilon,\delta}}g(\zeta)d\zeta=0\,,$$

where $\gamma_{\epsilon,\delta}$ is the keyhole shaped contour contained in $\tilde{D}$, as shown. Here, δ is the width of the corridor, and ϵ is both the radius of the small circular arc centred at z and the distance of the large circular arc from the boundary circle C.



If, for each fixed $\epsilon>0$, we let the width δ of the corridor go to zero, the contributions to the integral of the two sides of the corridor cancel due to the continuity of g, and we obtain

$$\lim_{\delta\to 0}\int_{\gamma_{\epsilon,\delta}}g(\zeta)d\zeta=\int_{C_\epsilon}g(\zeta)d\zeta-\int_{\Lambda_\epsilon}g(\zeta)d\zeta=0\,,$$

where C_ϵ is the circle in D, centred at the centre of D and at distance ϵ from C, and Λ_ϵ is the circle of radius ϵ centred at z, both with the positive orientation.

We now consider what happens as $\epsilon \to 0$. We have

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} g(\zeta) d\zeta = \int_C g(\zeta) d\zeta$$

due to the continuity of g on $\Omega \setminus \{z\}$. We also have

$$\lim_{\epsilon \to 0} \int_{\Lambda_\epsilon} g(\zeta) d\zeta = \lim_{\epsilon \to 0} \int_{\Lambda_\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta = 2\pi i f(z) \,,$$

similarly as in lectures, by the holomorphicity of f at z.

Thus we have

$$\int_C g(\zeta) d\zeta = \int_C \frac{f(\zeta)}{\zeta - z} d\zeta = 2\pi i f(z)$$

as required. □