COMPLEX LOGARITHMS

Let $z \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ be a non-zero complex number. We say that $\lambda \in \mathbb{C}$ is a *logarithm* of z if $e^\lambda = z$. (Note that there is no $\lambda \in \mathbb{C}$ such that $e^\lambda = 0$, and thus 0 does not have a logarithm.) If we write $z = |z|e^{i\theta}$, $\theta \in \mathbb{R}$, and $\lambda = x + iy$, we see that

$$z = |z|e^{i\theta} = e^\lambda = e^{x+iy} = e^x e^{iy}\,.$$

We have that $|z| = e^x$, so $x = \log|z|$, while $e^{i\theta} = e^{iy}$ implies that $y = \theta + 2k\pi$, $k \in \mathbb{Z}$. Thus x is uniquely determined, but there are infinitely many possible choices for y, with any two choices differing by an integer multiple of 2π. We see therefore that

$$e^\lambda = z \quad \text{if and only if} \quad \lambda = \log|z| + i\theta$$

where $\theta \in \mathbb{R}$ is any argument for z.

Let $\Omega \subset \mathbb{C}^*$ be an open set. A *logarithm* on Ω is a holomorphic function $\lambda : \Omega \to \mathbb{C}$ such that $\exp(\lambda(z)) = z$ for all $z \in \Omega$. (We will later see that this condition ensures any logarithm is holomorphic.) From the above discussion we see that we must always have $\mathrm{Re}(\lambda(z)) = \log|z|$. We also see that $\mathrm{Im}(\lambda(z))$ is a continuous choice of argument for the points $z \in \Omega$. However, depending on the set Ω, such a choice may or may not exist.

Theorem 1. *Let $\Omega \subset \mathbb{C}^*$ be an region. A function $\lambda : \Omega \to \mathbb{C}$ is a logarithm on Ω if and only if λ is a primitive for $1/z$ that satisfies $\exp(\lambda(c)) = c$ for some $c \in \Omega$.*

Proof. Let $\lambda : \Omega \to \mathbb{C}$ be a logarithm. Differentiating $\exp(\lambda(z)) = z$ gives $1 = \exp(\lambda(z))\lambda'(z) = z\lambda'(z)$ (since $\exp(\lambda(z)) = z$). Since $z \neq 0$ we obtain $\lambda'(z) = 1/z$, and clearly $\exp(\lambda(c)) = c$ for all $c \in \Omega$.

Conversely, suppose that $\lambda : \Omega \to \mathbb{C}$ is a holomorphic function such that $\lambda'(z) = 1/z$ and $\exp(\lambda(c)) = c$ for some $c \in \Omega$. Then

$$\frac{\partial}{\partial z}(z\exp(-\lambda(z))) = \exp(-\lambda(z)) + z\exp(-\lambda(z))(-\lambda'(z)) = 0\,,$$

so $z\exp(-\lambda(z))$ is a constant (since Ω is connected). Evaluating at $z = c$ we see that the constant is 1. Thus $\exp(\lambda(z)) = z$ for all $z \in \Omega$. $\square$

Corollary 1. *A region $\Omega \in \mathbb{C}^*$ has a logarithm if and only if the reciprocal function $1/z$ has a primitive on Ω.*

Proof. Let λ be a primitive for $1/z$ on Ω. Choose $c \in \Omega$ and set $\tilde{\lambda}(z) = \lambda(z) + K$ where $K \in \mathbb{C}$ is a constant chosen so that $e^K = ce^{-\lambda(c)}$. Then $\tilde{\lambda}$ is still a primitive for $1/z$ and $\exp(\tilde{\lambda}(c)) = c$. By the previous result $\tilde{\lambda}$ is a logarithm on Ω. $\square$

Corollary 2. *Every simply connected region $\Omega \in \mathbb{C}^*$ has a logarithm. Any two logarithms differ by an integer multiple of $2\pi i$.*

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Proof. Fix some point $z_0 \in \Omega$. Choose $\lambda_0 \in \mathbb{C}$ such that $e^{\lambda_0} = z_0$. Given $z \in \Omega$, define

$$\lambda(z) = \lambda_0 + \int_\gamma \frac{1}{\zeta} d\zeta$$

where γ is any path in Ω from z_0 to z.

Since Ω is simply connected, the integral is independent of path from z_0 to z (any two such paths are homotopic with endpoints fixed and therefore give the same integral). We see that λ is a primitive for $1/z$ by differentiating the integral in the same way as in the proof of the existence of primitives on an open disc (Stein and Shakarchi, Theorem 2.1, Chapter 2). We have $\exp(\lambda(z_0)) = \exp(\lambda_0) = z_0$, so λ is a logarithm on Ω.

Now suppose that λ_1 and λ_2 are both logarithms on Ω. Then $\exp(\lambda_1(z)) = \exp(\lambda_2(z))$ for all $z \in \Omega$. Thus $\exp(\lambda_1(z) - \lambda_2(z)) = 1$ for all $z \in \Omega$. Differentiating, $(\lambda_1'(z) - \lambda_2'(z)) \exp(\lambda_1(z) - \lambda_2(z)) = (\lambda_1'(z) - \lambda_2'(z)) = 0$, and $\lambda_1(z) - \lambda_2(z)$ is a constant. But $\exp(\lambda_1(z) - \lambda_2(z)) = 1$ implies that the constant must be $2k\pi i$ for some $k \in \mathbb{Z}$, and then $\lambda_1(z) = \lambda_2(z) + 2k\pi i$. ☐

Example 1. *There does not exist a logarithm on $\mathbb{C}^*$ because there is no primitive for $1/z$ on $\mathbb{C}^*$.*

Definition 1. *Consider the slit plane $\mathbb{C} \setminus \{z \in \mathbb{C} : \mathrm{Re}(z) \leq 0, \mathrm{Im}(z) = 0\}$ (the complex plane with the ray from 0 along the negative real axis removed). The* principal branch *of the logarithm is the function*

$$\mathrm{Log} : \mathbb{C} \setminus \{z \in \mathbb{C} : \mathrm{Re}(z) \leq 0, \mathrm{Im}(z) = 0\} \to \mathbb{C}$$

given by the formula

$$\mathrm{Log}(z) = \log|z| + i\theta \quad \textit{where} \quad z = |z|e^{i\theta} \quad \textit{and} \quad \theta \in (-\pi, \pi)\,.$$

Example 2. *There exist non-simply connected regions on which a logarithm exists. For example, let $\Omega = D_1(2) \setminus \{2\} \subset \mathbb{C}^*$ be the disc of radius 1, centred at 2, with the point at 2 removed. Then the restriction $\lambda = \mathrm{Log}\,|_\Omega$ is a logarithm on Ω.*

The rest of the material on complex logarithms can be found on page 100 of Stein and Shakarchi.