LAURENT SERIES

We say that a series $\sum_{n=-\infty}^{\infty} b_n$, $b_n \in \mathbb{C}$, *converges with sum* S if the series $\sum_{n=-\infty}^{-1} b_n$ and $\sum_{n=0}^{\infty} b_n$ both converge, and $\sum_{n=-\infty}^{-1} b_n + \sum_{n=0}^{\infty} b_n = S$.

A series of the form $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is called a *Laurent series.* The *principal part* of the Laurent series is the series $\sum_{n=-\infty}^{-1} a_n(z-z_0)^n$.

Consider the part $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ of the Laurent series. This is a power series in $z-z_0$, hence the largest open set on which it converges is determined by $|z-z_0| < r_2$ for some $r_2 \in [0,\infty]$, that is, an open disc centred at z_0 of radius r_2 (all of $\mathbb{C}$ if $r_2 = \infty$, and the empty set if $r_2 = 0$). Now consider the principal part $\sum_{n=-\infty}^{-1} a_n(z-z_0)^n$ of the Laurent series. Changing the summation variable to $m = -n$ we see that this is actually a power series in $\frac{1}{z-z_0}$. Thus the largest open set on which it converges is determined by $|\frac{1}{z-z_0}| < s$ for some $s \in [0,\infty]$, that is, $|z-z_0| > r_1 = 1/s$, where $r_1 = 0$ if $s = \infty$, $r_1 = \infty$ if $s = 0$, and $r_1 = -1$ if there is no principal part (we explain this choice in a moment). Taking the intersection of these sets, we see that the largest open set on which the Laurent series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ converges is an *annulus* $r_1 < |z-z_0| < r_2$ (the empty set if $r_1 \geq r_2$). Note that if the Laurent series has no principal part then $r_1 = -1$ and the series converges on $-1 < |z-z_0| < r_2$, that is, on the open disc $|z-z_0| < r_2$.

Example 1. *$e^{1/z}$ has a Laurent series expansion centred at 0,*

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=-\infty}^{0} \frac{z^n}{(-n)!}.$$

Example 2. *Let $f : \mathbb{C} \setminus \{1\} \to \mathbb{C}$, $f(z) = \frac{1}{1-z}$. If $|z| < 1$, $f(z) = \sum_{n=0}^{\infty} z^n$. If $|z| > 1$, then $\frac{1}{|z|} < 1$, hence*

$$f(z) = \frac{1}{1-z} = -\frac{1}{z} \cdot \frac{1}{(1-\frac{1}{z})} = -\frac{1}{z}\sum_{n=0}^{\infty}(\frac{1}{z})^n = -\sum_{n=-\infty}^{-1} z^n = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots$$

is the Laurent series for $\frac{1}{1-z}$ on the annulus $|z| > 1$.

Date: September 30, 2014.

Theorem 1. *Let f be holomorphic on the annulus A given by $r_1 < |z - z_0| < r_2$, where $r_1, r_2 \in [0, \infty]$. Then there is a unique Laurent series $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ such that*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad \text{for all} \quad z \in A\,.$$

For every $r \in (r_1, r_2)$,

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

for all $n \in \mathbb{Z}$.

Corollary 1. *If f has an essential singularity at z_0 then f has a Laurent series expansion at z_0 with infinite principal part.*