Definition Let $f(z) = \dfrac{az+b}{cz+d}$, $a,b,c,d \in \mathbb{C}$, $ad-bc \neq 0$.

Then f is called a fractional linear transformation, or Möbius transformation.

Note: The condition $ad-bc \neq 0$ ensures the following conditions hold:

- neither $az+b$, nor $cz+d$ ~~does not~~ vanish identically.
- ~~$cz+d$ " " " "~~
- ~~not both~~ a and c cannot both be zero (in which case f would be constant)
- b and d cannot be both zero (f would be constant
- in general, the denominator cannot be a constant multiple of the numerator, ~~that is, $az+b$ and $cz+d$ do not have common factors.~~

Indeed, if $k(az+b) = cz+d$ for some $k \in \mathbb{C}$, then equating coefficients $\Rightarrow$ $ka = c$, $kb = d$.

and $ad - bc = a(kb) - b(ka) = 0$.

Thus $ad-bc \neq 0$ ensures $az+b$ and $cz+d$ do not have a common factor, and therefore $f(z)$ is a well defined non-constant hol. function

$\mathbb{P} \to \mathbb{P}$.

Let $GL_2(\mathbb{C})$ denote the group of 2×2 matrices M with complex entries and non-zero determinant.

$$GL_2(\mathbb{C}) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{C},\ \det(M) = ad-bc \neq 0 \right\}$$

(called general linear group).

As before, $M \in GL_2(\mathbb{C})$

defines $f_M : \mathbb{P} \to \mathbb{P}$, $f_M(z) = \dfrac{az+b}{cz+d}$.

Recall that $f_M(\infty) = a/c$ if $c \neq 0$,

$f_M(\infty) = \infty$ if $c = 0$,

$f_M(-d/c) = \infty$ if $c \neq 0$.

Theorem Every $f \in \mathrm{Aut}(\mathbb{P})$ can be written in the form $f = f_M$ for some $M \in GL_2(\mathbb{C})$. Conversely, ~~every~~ $f_M \in \mathrm{Aut}(\mathbb{P})$ $\forall\, M \in GL_2(\mathbb{C})$.

Proof:

As before, if $M, N \in GL_2(\mathbb{C})$ then $f_M \circ f_N = f_{M \cdot N}$. Thus each $f_M \in \mathrm{Aut}(\mathbb{P})$ with inverse $f_{M^{-1}}$.

Let $f \in \mathrm{Aut}(\mathbb{P})$. By our previous discussion, f must be rational, and non-constant, with numerator and denominator of degree at most 1. This is captured precisely by saying that $f(z) = \dfrac{az+b}{cz+d}$, $a,b,c,d \in \mathbb{C}$, $ad-bc \neq 0$.

So $f = f_M$ for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. ▨ Thus $\mathrm{Aut}(\mathbb{P}) = \left\{ z \mapsto \dfrac{az+b}{cz+d} : a,b,c,d \in \mathbb{C}, ad-bc \neq 0 \right\}$

Let $M \in GL_2(\mathbb{C})$, and $\lambda \in \mathbb{C}^*$.

Note; ~~Let~~ We have $f(z) = f_M(z) = \dfrac{az+b}{cz+d} = \dfrac{\lambda az + \lambda b}{\lambda cz + \lambda d} = f_{\lambda M}(z)$

and $(\lambda a)(\lambda d) - (\lambda c)(\lambda b) = \lambda^2(ad-bc) \neq 0$, so $\lambda M \in GL_2(\mathbb{C})$ defines the same linear transformation as M, $\forall \lambda \in \mathbb{C}^*$.

So we are free to choose λ so that $\det \lambda M = 1$, that is, we can find $M \in SL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{C}, \det M = 1 \right\}$ s.t. $f = f_M$. As before, $M, -M$ both in $SL_2(\mathbb{C})$ defines the same $f_M = f_{-M}$, so we identify them to get $PSL_2(\mathbb{C})$.

We have $\mathrm{Aut}(\mathbb{P}) \cong PSL_2(\mathbb{C}) \cong \dfrac{SL_2(\mathbb{C})}{\{\pm I\}} \cong \dfrac{GL_2(\mathbb{C})}{\mathbb{C}^*}$.

The group $\mathrm{Aut}(\mathbb{P})$ //

Properties of Möbius transformations

Lemma: Every Möbius transformation is the composition of the four elementary maps (each a Möbius trans.):

(a) translations $z \mapsto z + z_0, \quad z_0 \in \mathbb{C}$
(b) dilations $z \mapsto \lambda z, \quad \lambda > 0, \ \lambda \in \mathbb{R}$
(c) rotations $z \mapsto e^{i\theta} z, \quad \theta \in \mathbb{R}$
(d) inversion $z \mapsto 1/z$

Proof: Let $f \in \mathrm{Aut}(\mathbb{P})$, $f(z) = \dfrac{az+b}{cz+d}$. If $c = 0$ then $f(z) = \frac{a}{d} z + \frac{b}{d}$, which is a composition of a scaling and rotation, followed by a translation.
If $c \neq 0$, write $f(z) = \dfrac{bc - ad}{c^2} \cdot \dfrac{1}{z + d/c} + \dfrac{a}{c}$

which is a translation, inversion, scaling, rotation, and translation (in that order). ∎

Fact:
- Let $S \subset \mathbb{C}$ be a circle. Using the correspondence $\mathbb{C} \leftrightarrow \mathbb{P} \setminus \{\infty\}$ given by stereographic projection, S maps to a circle $S' \subset \mathbb{P} \setminus \{\infty\}$. Conversely, any circle S' in $\mathbb{P}$ not through ∞ maps to a circle S in $\mathbb{C}$.
- Let $L \subset \mathbb{C}$ be a straight line. Using $\mathbb{C} \leftrightarrow \mathbb{P} \setminus \{\infty\}$, L maps to $L' \subset \mathbb{P}$, a circle ~~[illegible]~~ through ∞, with the point at infinity removed. Conversely, if $L' \subset \mathbb{P}$ is a circle through ∞ with ∞ removed then stereographic proj. of L' ~~[illegible]~~ gives a line L in $\mathbb{C}$.
  Adding the pt at ∞: line in $\mathbb{C}$ with ∞ $L \cup \{\infty\}$ $\leftrightarrow$ $L' \cup \{\infty\}$ circle in $\mathbb{P}$ through ∞.

Summary:

circles in $\mathbb{C}$ $\longleftrightarrow$ circles in $\mathbb{P}$ not through ∞

lines in $\mathbb{C}$ $\longleftrightarrow$ circles in $\mathbb{P}$ through ∞.

~~Definition: We call the set~~

Prop. ~~Lemma~~: ~~Fact:~~ Möbius transformations preserve circles in $\mathbb{P}$. That is, a Möbius transformation maps circles in $\mathbb{P}$ to circles in $\mathbb{P}$.

Proof: Let S be a circle in $\mathbb{C}$. Clearly, translations, dilations and rotations all map S to another circle in $\mathbb{C}$. Similarly, if L is a line in $\mathbb{C}$ then translations, dilations, rotations all map L to another line L' and also map ∞ to ∞. So these three types of elementary maps preserve circles in $\mathbb{P}$.

Let $|z - z_0| = r$ be a circle in $\mathbb{C}$, and let $w = \frac{1}{z}$. We get $|z - z_0|^2 = r^2$

$$\Rightarrow (z - z_0)(\bar{z} - \bar{z}_0) - r^2 = 0$$

$$\Rightarrow |z|^2 - 2\,\mathrm{Re}(\bar{z} z_0) + |z_0|^2 - r^2 = 0$$

$$\Rightarrow \quad \frac{1}{|w|^2} - 2\frac{\mathrm{Re}(w z_0)}{|w|^2} + |z_0|^2 - r^2 = 0. \qquad \left(\bar z = \frac{1}{\bar w} = \frac{w}{|w|^2}\right)$$

If $|z_0| = r$, get $2\mathrm{Re}(w z_0) = 1$, letting $w = u + iv$, $z_0 = x_0 + iy_0$,
$\Rightarrow 2(u x_0 - v y_0) = 1$, a line in $\mathbb{C}$. ($|z_0| = r \Rightarrow$ the circle passes through the origin).

Otherwise, get $1 - 2\mathrm{Re}(w z_0) + (|z_0|^2 - r^2)|w|^2 = 0$

$$\Rightarrow \quad |w|^2 - \frac{2\mathrm{Re}(w z_0)}{|z_0|^2 - r^2} + \frac{(|z_0|^2 - r^2)}{(|z_0|^2 - r^2)^2} = 0$$

$$\Rightarrow \quad \left| w - \frac{\bar z_0}{|z_0|^2 - r^2} \right|^2 - \frac{r^2}{(|z_0|^2 - r^2)^2} = 0$$

which is the equation of a circle in w.

Otherwise, if we have a line $L: 2\mathrm{Re}(z \bar z_0) = a$, $a \in \mathbb{R}$, then $w = 1/z$ gives $2\mathrm{Re}(w z_0) = a|w|^2$. If $a = 0$ (so that L is through origin) then $2\mathrm{Re}(w z_0) = 0$ is another line (through the origin).

Otherwise, get $w z_0 + \bar w \bar z_0 - a w \bar w = 0 \Rightarrow$

$$w\bar w - w z_0/a - \bar w \bar z_0/a + \frac{|z_0|^2}{a^2} - |z_0|^2/a^2 = 0$$

$$\Rightarrow \quad \left| w - \bar z_0/a \right|^2 = \left(\left| \frac{z_0}{a} \right| \right)^2, \text{ a circle in } \mathbb{C}.$$

Thus all 4 elementary maps preserve circles in $\mathbb{P}$. Hence every composition of elementary maps also preserves circles in $\mathbb{P}$, and this gives all Möbius transformations by the previous lemma. ∎

How many fixed points can a Möbius transformation have?

Let $f \in \mathrm{Aut}(\mathbb{P})$, $f(z) = \frac{az+b}{cz+d}$. Suppose z is fixed by f,

$$f(z) = z = \frac{az+b}{cz+d} \Rightarrow \quad cz^2 + dz = az + b \Rightarrow$$
$$cz^2 + (d-a)z - b = 0.$$

If $c \neq 0$ this is a quadratic and has either 1 or 2 solutions in $\mathbb{C}$. Also note that ∞ is not fixed.

If $c = 0$ and $d \neq a$ then this is linear and has 1 solution in $\mathbb{C}$. But then $f(z) = \frac{a}{d} z + \frac{b}{d}$

satisfies $f(\infty) = \infty$, and ∞ is also a fixed point.
If $c = 0$ and $a = d$, then $f(z) = z + b/d$, if ~~b/d~~ $b \neq 0$
then $f(\infty) = \infty$ is the only fixed point of f.
(Otherwise, $f(z) = z$ is the identity map and fixes every point of $\mathbb{P}$).
Thus, every $f \in \mathrm{Aut}(\mathbb{P})$, $f \neq id_{\mathbb{P}}$, has ~~fixed~~ either 1 or 2 fixed points in $\mathbb{P}$.

Lemma A Möbius transformation is completely determined its action on three distinct points.

Proof: Let $S, T \in \mathrm{Aut}(\mathbb{P})$ and suppose $\exists$ pts $a, b, c \in \mathbb{P}$ s.t. $S(a) = T(a) =: \alpha$ $S(b) = T(b) =: \beta$ $S(c) = T(c) =: \gamma$. But then $S \circ T^{-1} \in \mathrm{Aut}(\mathbb{P})$ is a Möbius transformation, and
$S \circ T^{-1}(\alpha) = S(a) = \alpha$, similarly $S \circ T^{-1}(\beta) = \beta$,
$S \circ T^{-1}(\gamma) = \gamma$. So $S \circ T^{-1}$ has 3 ~~distinct~~ fixed points hence $S \circ T^{-1} = id \Rightarrow S = T$. $\blacksquare$

Let z_1, z_2, z_3 be distinct points in $\mathbb{P}$. Define $S \in \mathrm{Aut}(\mathbb{P})$ by

$$S(z) = \begin{cases} \dfrac{(z - z_1)}{(z - z_3)} \cdot \dfrac{(z_2 - z_3)}{(z_2 - z_1)} & \text{if } z_1, z_2, z_3 \in \mathbb{C}, \\[2ex] \dfrac{(z_2 - z_3)}{(z - z_3)} & \text{if } z_1 = \infty \\[2ex] \dfrac{(z - z_1)}{(z - z_3)} & \text{if } z_2 = \infty \\[2ex] \dfrac{(z - z_1)}{(z_2 - z_1)} & \text{if } z_3 = \infty. \end{cases}$$

Then $S(z_1) = 0$, $S(z_2) = 1$, $S(z_3) = \infty$, and S is the only Möbius transformation with this property.

Definition: Let $z_0, z_1, z_2, z_3 \in \mathbb{P}$. Their cross-ratio is defined as

$$[z_0 : z_1 : z_2 : z_3] := \frac{z_0 - z_1}{z_0 - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

$= S(z_0)$, where $S \in \mathrm{Aut}\,\mathbb{P}$ satisfies $S(z_1) = 0$, $S(z_2) = 1$, $S(z_3) = \infty$.

ex: $[z_1; z_1; z_2 : z_3] = 0 \qquad [z_2; z_1; z_2; z_3] = 1$

$[z; 0; 1; \infty] = z$

$[z; \infty; 1; 0] = 1/z$.

Prop: ~~The cross ratio~~ If $z_1, z_2, z_3 \in \mathbb{P}$ are distinct pts and $T \in \mathrm{Aut}(\mathbb{P})$ is a Möbius trans. then

$$[z_0; z_1; z_2, z_3] = [T(z_0) : T(z_1) : T(z_2) : T(z_3)]$$

for all $z_0 \in \mathbb{P}$. (Möbius trans. preserve cross ratios).

Proof: Let $S(z) = [z; z_1; z_2; z_3]$, then $S \in \mathrm{Aut}(\mathbb{P})$.
Letting $M = S \circ T^{-1}$ we have $M(T(z_1)) = 0$, $M(T(z_2)) = 1$, $M(T(z_3)) = \infty$, hence
$M(w) = S \circ T^{-1}(w) = [w : T(z_1) : T(z_2) : T(z_3)]$.
Letting $w = T(z_0)$, we get

$$S \circ T^{-1}(w) = S \circ T^{-1}(T(z_0)) = S(z_0) = [T(z_0) : T(z_1) : T(z_2) : T(z_3)]$$

∎

~~Prop: Four distinct points in $\mathbb{P}$ lie on a circle iff their cross ratio is real.~~
~~Proof: Let $z_0, z_1, z_2, z_3 \in \mathbb{P}$.~~

Prop: Let $z_1, z_2, z_3 \in \mathbb{P}$ be distinct, and $w_1, w_2, w_3 \in \mathbb{P}$ be distinct. There is a unique $S \in \mathrm{Aut}(\mathbb{P})$ s.t. $S(z_j) = w_j$, $j = 1, 2, 3$.

Proof: Let $T(z) = [z : z_1; z_2 : z_3]$, $R(z) = [w : w_1 : w_2 : w_3]$. Then $S = R^{-1} \circ T \in \mathrm{Aut}(\mathbb{P})$ has the desired property. By earlier lemma, S is unique. ∎

Recall: 3 points in $\mathbb{C}$ determine a circle or a line. So 3 points in $\mathbb{P}$ determine a circle in $\mathbb{P}$.

Prop: Given two circles S, S' in $\mathbb{P}$, there is a Möbius transformation T taking S to S'.

Proof: Let $z_1, z_2, z_3 \in S$, and $w_1, w_2, w_3 \in S'$, and let $T \in \mathrm{Aut}(\mathbb{P})$, $T(z_1) = w_1$, $T(z_2) = w_2$, $T(z_3) = w_3$. Then $T(S)$ is a circle passing through w_1, w_2, w_3, hence $T(S) = S'$. ∎