UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT2410 — Introduction to Complex Analysis

Day of examination: Monday December 12th 2016

Examination hours: 9:00 – 13:00

This problem set consists of 5 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Determine the image of the open disk $|z| < 1$ under the fractional linear transformation

$$f(z) = \frac{2}{z - i}.$$

Solution: Let D denote the disk $|z| < 1$. We know that $f(\partial D)$ is a circle in the Riemann sphere. Because $f(i) = \infty$, that circle has the form $L \cup \{\infty\}$ for some line L in the complex plane. To determine L we compute

$$f(-i) = i, \quad f(1) = 1 + i.$$

It follows that L is the line $\mathrm{Im}(z) = 1$. Since $\mathrm{Im}(f(0)) = 2 > 1$, we see that $f(D)$ is the half-plane $\mathrm{Im}(z) > 1$.

Problem 2

Given a complex number $w \neq 0$ consider the function

$$f(z) = \frac{z^9}{z^5 - w^5}.$$

a

Find the Laurent series of $f(z)$ in the region $|z| > |w|$.

(Continued on page 2.)

Solution: For $|z| > |w|$ one has

$$\begin{aligned} f(z) &= \frac{z^4}{1 - (\frac{w}{z})^5} \\ &= z^4 \sum_{k=0}^{\infty} \left(\frac{w}{z}\right)^{5k} \\ &= z^4 \sum_{n=-\infty}^{0} \left(\frac{z}{w}\right)^{5n} \\ &= \sum_{n=-\infty}^{0} \frac{z^{5n+4}}{w^{5n}} \\ &= z^4 + \frac{w^5}{z} + \frac{w^{10}}{z^6} + \cdots . \end{aligned}$$

b

Let $r > |w|$. Use the result of **a** to compute the line integral

$$\int_{|z|=r} f(z)\, dz.$$

Solution: The general formula for the coefficients of a Laurent series yields

$$\int_{|z|=r} f(z)\, dz = 2w^5 \pi i.$$

Problem 3

Let $f(z)$ be a non-constant analytic function in $\mathbb{C} \setminus \{0\}$ such that

$$z^3 f(z) \to 0 \quad \text{as } z \to 0.$$

a

What kind of isolated singularity can $f(z)$ have at 0 ?

Solution: By Riemann's removable singularities theorem there is an entire function $g(z)$ such that $g(z) = z^2 f(z)$ for $z \neq 0$. Hence

$$f(z) = \frac{g(z)}{z^2}$$

has at most a pole of order 2 at 0.

b

Suppose in addition that

$$f(z) \to 0 \quad \text{as } z \to \infty.$$

(Continued on page 3.)

Show that there exist constants $a, b \in \mathbb{C}$ such that

$$f(z) = \frac{a}{z} + \frac{b}{z^2}.$$

Solution: Let

$$P(z) = \frac{a}{z} + \frac{b}{z^2}$$

be the principal part of $f(z)$ at 0. Then $h(z) = f(z) - P(z)$ has a removable singularity at 0 and satisfies

$$h(z) \to 0 \quad \text{as } z \to \infty.$$

By Liouville's theorem, $h(z)$ must be identically zero, so $f(z) = P(z)$.

Problem 4

Let D be a bounded domain and ∂D its boundary. Let $f(z)$ be a continuous function on the bounded closed set $D \cup \partial D$ such that $f(z)$ is analytic on D and real-valued on ∂D. Show that $f(z)$ is constant. *Hint:* Apply the maximum principle to the imaginary part of $f(z)$.

Solution: Let $f = u + vi$, where u, v are real-valued. Then v is harmonic in D and vanishes on ∂D. By the maximum principle, v must be identically zero. The Cauchy-Riemann equations says that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$$

in D. Therefore, $f = u$ is constant.

Problem 5

Consider the function

$$f(z) = \frac{e^{iz} - 1}{z + z^3}.$$

a

Classify all isolated singularities of $f(z)$ in the complex plane and compute the residue at each pole.

Solution: The numerator $e^{iz} - 1$ has a simple zero at $2\pi k$ for every integer k, whereas the denominator

$$z + z^3 = z(z - i)(z + i)$$

has simple zeros at $0, i, -i$. Hence, $f(z)$ has a removable singularity at 0 and simple poles at $\pm i$. As for the residues,

$$\operatorname{Res}[f(z), i] = \lim_{z \to i} (z - i) f(z) = \left. \frac{e^{iz} - 1}{z(z + i)} \right|_{z=i} = \frac{1}{2}(1 - e^{-1}),$$

$$\operatorname{Res}[f(z), -i] = \left. \frac{e^{iz} - 1}{z(z - i)} \right|_{z=-i} = \frac{1}{2}(1 - e).$$

(Continued on page 4.)

b

What kind of singularity does $f(z)$ have at ∞ ?

Solution: For $t > 1$ we have

$$|f(t)| \leq \frac{2}{t+t^3} \to 0 \quad \text{as } t \to \infty,$$
$$|f(-it)| = \frac{e^t - 1}{t^3 - t} \to \infty \quad \text{as } t \to \infty.$$

Therefore, $f(z)$ cannot have a removable singularity or a pole at ∞, so it has an essential singularity there.

c

For $r > 1$ consider the curve

$$\gamma_r(t) = re^{it}, \quad 0 \leq t \leq \pi.$$

Show that

$$\int_{\gamma_r} f(z)\, dz \to 0 \quad \text{as } r \to \infty.$$

Solution: For any $z \in \mathbb{C}$ one has $|e^{iz}| = e^{-\mathrm{Im}(z)}$. If $|z| = r$, $\mathrm{Im}(z) \geq 0$ this gives

$$|f(z)| \leq \frac{2}{r^3 - r}.$$

The ML-estimate gives

$$\left| \int_{\gamma_r} f(z)\, dz \right| \leq \pi r \cdot \frac{2}{r^3 - r} = \frac{2\pi}{r^2 - 1} \to 0 \quad \text{as } r \to \infty.$$

d

Apply the residue theorem in the domain

$$D_r = \{z \in \mathbb{C} : |z| < r,\ \mathrm{Im}(z) > 0\}$$

together with the results of **a** and **c** to compute the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x + x^3}\, dx.$$

Solution: For $r > 1$ the residue theorem gives

$$2\pi i \operatorname{Res}[f(z), i] = \int_{\partial D_r} f(z)\, dz = \int_{-r}^{r} f(x)\, dx + \int_{\gamma_r} f(z)\, dz.$$

Letting $r \to \infty$ we get

$$\pi i(1 - e^{-1}) = \int_{-\infty}^{\infty} f(x)\, dx.$$

Taking imaginary parts we obtain

$$\int_{-\infty}^{\infty} \frac{\sin x}{x + x^3}\, dx = \pi(1 - e^{-1}).$$

(Continued on page 5.)

Problem 6

a

Determine the number of zeros, counting multiplicity, of the function

$$f(z) = e^z + 5z^7 - 2$$

in the square

$$R = \{z \in \mathbb{C} : |\mathrm{Re}(z)| < 1, \ |\mathrm{Im}(z)| < 1\}.$$

Solution: For any $z \in \partial D$ we have

$$|e^z| = e^{\mathrm{Re}(z)} \leq e < 3,$$

so

$$|e^z - 2| \leq |e^z| + 2 < 5 \leq |5z^7|.$$

By Rouché's theorem, the functions $f(z)$ and $g(z) = 5z^7$ have the same number of zeros in D, counting multiplicity. Since 0 is the only zero of $g(z)$ and this zero has order 7, we conclude that $f(z)$ has 7 zeros in D counting multiplicity.

b

Show that if z_0 is a zero of $f(z)$ of order $k \geq 1$ then $\bar{z}_0$ is also a zero of $f(z)$ of order k.

Solution: Because $f(z)$ is given by a power series centred at the origin where all coefficients are real, we have

$$\overline{f(z)} = f(\bar{z}).$$

The power series representation of $f(z)$ at z_0 has the form

$$f(z) = \sum_{n=k}^{\infty} a_n (z - z_0)^n$$

with $a_k \neq 0$. Therefore,

$$f(z) = \overline{f(\bar{z})} = \sum_{n=k}^{\infty} \bar{a}_n (z - \bar{z}_0)^n,$$

so $\bar{z}_0$ is also a zero of $f(z)$ of order k.

c

How many of the zeros found in part **a** are real, and how many have positive imaginary part?

Solution: Because

$$f(-1) = e^{-1} - 7 < 0, \quad f(1) = e + 3 > 0$$

and $f'(x) > 0$ for x real, we see that $f(z)$ has exactly one real zero in D, and this zero is simple. Consequently, the number of zeros of $f(z)$ in D with positive imaginary part is $(7-1)/2 = 3$, counting multiplicity.

END