MAT 2440 Solutions

May 30, 2016

Problem 1.

(**a**) We shall first find the eigenvalues of

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}.$$

Hence we calculate

$$\begin{aligned} |a - \lambda I| &= \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & -1 \\ -1 & 1 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 0 & 1-\lambda \\ -1 & 1 \end{vmatrix} \\ &= (1-\lambda)[1+\lambda^2 - 2\lambda + 1] + 1 - \lambda = (1-\lambda)^3 \end{aligned}$$

Hence $\lambda = 1$ has multiplicity three, and is the only eigenvalue. We proceed to find the eigenspace:

$$A - I = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Hence if $\mathbf{v}_1 = [x, y, z]^T$ is an eigenvector, then $z = 0, -x + y = 0$, and the eigenspace is spanned by the single vector $\mathbf{v}_1 = [1, 1, 0]^T$, and two linearly independent generalized eigenvectors are needed in addition to $\mathbf{v}_1$. Now, as is easily seen, $(A - I)^3 = 0$, so that any vector $\mathbf{v}_3 \neq \mathbf{0}$ that is linearly independent of $\mathbf{v}_1$ may be tried. We take $\mathbf{v}_3 = [1, 0, 0]^T$. Then

$$(A - I)\mathbf{v_3} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \mathbf{v_2}$$

Finally,

$$(A - I)\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \mathbf{v_1}$$

Using the above generalized eigenvectors we find the general solution

$$\mathbf{x}(t) = ae^t\mathbf{v_1} + be^t(t\mathbf{v_1} + \mathbf{v_2}) + ce^t(\frac{1}{2}t^2\mathbf{v_1} + t\mathbf{v_2} + \mathbf{v_3})$$

Hence

$$\mathbf{x}(t) = e^t \begin{bmatrix} a + bt + \frac{1}{2}t^2c + c \\ a + bt + \frac{1}{2}t^2c \\ -b - ct \end{bmatrix}$$

(**b**) First, a fundamental matrix $\Phi(t)$ of A is given by the column vectors

$$e^t\mathbf{v_1}, e^t(t\mathbf{v_1} + \mathbf{v_2}), e^t(\frac{1}{2}t^2\mathbf{v_1} + t\mathbf{v_2} + \mathbf{v_3})$$

Moreover,

$$\begin{aligned} e^{tA} &= \Phi(t)\Phi(0)^{-1} \\ &= e^t \begin{bmatrix} 1 & t & 1 + \frac{1}{2}t^2 \\ 1 & t & \frac{1}{2}t^2 \\ 0 & -1 & -t \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \\ &= e^t \begin{bmatrix} 1 + \frac{1}{2}t^2 & -\frac{1}{2}t^2 & -t \\ \frac{1}{2}t^2 & 1 - \frac{1}{2}t^2 & -t \\ -t & t & 1 \end{bmatrix} \end{aligned}$$

Problem 2

(**a**) The critical points occur for
$y - xy^2 = y(1 - xy) = 0$ and $x - x^2y = x(1 - xy) = 0$. Hence $x = y = 0$ or $xy = 1$. Consequently, the critical points consist of $(0, 0)$ and all the points on the hyperbola $xy = 1$.

(**b**) The critical points on the hyperbola $xy = 1$ are not isolated since any neighborhood of such points contains other points on the hyperbola. Hence the system fails (by definition) to be almost linear at such critical points. Clearly, $(0, 0)$ is an isolated critical point: For example, the neighborhood $\{(x, y) : x^2 + y^2 < 1\}$ of $(0, 0)$ contains no points on the hyperbola $xy = 1$, hence contains no other critical points.

Moreover, at the critical point $(0, 0)$ the system is almost linear since polynomial functions are C^1 so that the 2nd order error terms of Taylor's formula in two variables tend to zero faster than $\sqrt{x^2+y^2}$ (that is,

$$(*) \qquad \frac{xy^2}{\sqrt{x^2+y^2}} \to 0 \text{ and } \frac{-x^2y}{\sqrt{x^2+y^2}} \to 0 \quad \text{as } \sqrt{x^2+y^2} \to 0)$$

and, <u>in addition</u>, the matrix $A = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}$ of the linear part has a determinant different from 0. (We may also verify $(*)$ directly by introducing polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. Then

$$|\frac{xy^2}{\sqrt{x^2+y^2}}| = |r\cos\theta\sin^2\theta| \le r \to 0 \text{ as } r \to 0,$$

and

$$|\frac{x^2y}{\sqrt{x^2+y^2}}| = |r\cos^2\theta\sin\theta| \le r \to 0 \text{ as } r \to 0.)$$

In order to determine the type of the critical point $(0,0)$, we find the eigenvalues of the matrix A of the linearized system $\dot{x} = -y$, $\dot{y} = 4x$. The eigenvalues are the purely imaginary numbers $\pm 2i$. Hence $(0,0)$ is a center of the linearized system. It is a center or a (asymptotically stable or unstable) spiral point of <u>the nonlinear system</u>. Further analysis is required to decide exactly which.

(**c**) To find an equation of the solution curves to the given system, we observe that

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{4x(1-xy)}{y(xy-1)} = -\frac{4x}{y} \qquad (xy \neq 1),$$

so that

$$y\,\mathrm{d}y = -4x\,\mathrm{d}x, \quad 2x^2 + \frac{1}{2}y^2 = k, \text{or } \frac{x^2}{C^2} + \frac{y^2}{(2C)^2} = 1,\ C^2 = \frac{1}{2}k \ge 0$$

which represent ellipses centered at the origin. In particular, we may conclude that $(0,0)$ is a (stable) center.

Alternative: In (1), multiply the first equation with $4x$ and multiply the second with y. Add the resulting equations to obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}(2x^2 + \frac{1}{2}y^2) = 4x\dot{x} + y\dot{y} = 0,$$

hence we find the equations $x^2 + \frac{1}{4}y^2 = C^2$ which represent the same ellipses as we found above (in addition, we see that the condition $xy \neq 1$ of the first method is unnecessary).

Problem 3

(**a**) We will solve

$$\max \int_0^2 (x-u)\,\mathrm{d}t,\ \dot{x} = x+u,\ x(0) = 0,\ x(2) \text{ is free}, u(t) \in [0,1] \text{for all } t \in [0,2].$$

The Hamiltonian of this (normal) problem is

$$H = H(t,x,u,p) = x - u + p(x+u) = (p+1)x + (p-1)u,$$

Which is a linear function of u (of first degree in u). Now for any optimal pair (x^*, u^*), $u = u^*(t)$ must maximize the function $u \mapsto (1+p(t))x^*(t) + (p(t)-1)u$, for any fixed t in the integration interval. Hence we must have

$$u^*(t) = 1,\ \text{if } p(t) > 1,\ \text{and } u^*(t) = 0 \text{ if } p(t) < 1,\ u^*(t) \text{ undetermined if } p(t) = 1.$$

Further, if $x = x^*, u = u^*$ form an optimal pair, then by the Maximum Principle there is a continuous and piecewise C^1-function p such that

$$\frac{\partial H^*}{\partial x} = -\dot{p},$$

except at the discontinuity points of u^*. That is, $\dot{p}+p = -1$, $p(t) = ae^{-t} - 1$, where $0 = p(2) = ae^{-2} - 1$ Hence $a = e^2$ and $p(t) = e^{2-t} - 1$. Therefore,

$$p(t) > 1 \iff e^{2-t} > 2 \iff 2 - t > \ln 2 \iff t < 2 - \ln 2.$$

From the state equation $\dot{x} - x = u$, we find $x^*(t) = be^t - 1$ on $[0, 2-\ln 2]$ where $x^*(0) = b - 1 = 0$, yields $x^*(t) = e^t - 1$, $u^*(t) = 1$, on $[0, 2-\ln 2]$. On the interval $[2-\ln 2, 2]$ we find $x^*(t) = Ce^t$ and since x^* must be continuous, we find

$$x^*(2-\ln 2) = Ce^{2-\ln 2} = \frac{1}{2}Ce^2,\ \text{and } x^*(2-\ln 2) = e^{2-\ln 2} - 1 = \frac{1}{2}e^2 - 1.$$

Hence $C = 2(\frac{1}{2}e^2 - 1)/e^2 = 1 - 2/e^2$, and $x^*(t) = (1-2/e^2)e^t$, $u^*(t) = 0$, on $[2-\ln 2, 2]$.

Finally, for t and p fixed, the Hamiltonian $H(t,x,u,p)$ is linear as a function of (x,u), hence is concave (and convex) (as an alternative, this also follows from the 2nd derivative test). The above pair (x^*, u^*) is therefore optimal by Mangasarian's Theorem.

(**b**) This is the same problem as in (*a*) but with the new endpoint condition $x(2) = \frac{1}{2}e^2$ We find the same conditions on u^* and the same differential equations of x^* and p as before.

Notice first that we cannot have $p(t) < 1$ on all of $(0, 2)$, since this would imply that $x^*(t) = Ce^t$ for all t in $[0, 2]$, which would contradict the endpoint conditions. A similar argument shows that the inequality $p(t) > 1$ does not hold on all of $(0, 2)$. Hence $p(t) - 1$ attains both positive and negative values on the interval. Since p is continuous, there exists a point s in $(0, 2)$ such that $p(s) = 1$. Then $p(s) = ae^{-s} - 1 = 1$, hence $a = 2e^s$, and

$$p(t) = 2e^s e^{-t} - 1, \quad t \in [0, 2].$$

Now $2e^t$ is strictly increasing. Hence,

$$t \in (s, 2] \Longrightarrow 2e^s e^{-t} < 2e^t e^{-t} = 2 \Longrightarrow p(t) < 1,$$

and

$$t \in [0, s) \Longrightarrow 2e^s e^{-t} > 2e^t e^{-t} = 2 \Longrightarrow p(t) > 1.$$

We have shown that $p(t) > 1$ on $[0, s)$ and $p(t) < 1$ on $(s, 2]$. Hence we find from the endpoint conditions,

$$x^*(0) = b - 1, b = 1 \quad \text{ and } x^*(2) = ce^2, c = \frac{1}{2}.$$

Since x^* is continuous at $t = s$,

$$x^*(s) = e^s - 1 = \frac{1}{2}e^s \Longrightarrow s = \ln 2.$$

In particular, $p(t) = 4e^{-t} - 1$ (notice that $p(\ln 2) = 1$ and that p is strictly decreasing) and

$$u^*(t) = 1,\ x^*(t) = e^t - 1,\ t \in [0, \ln 2]$$

and

$$u^*(t) = 0,\ x^*(t) = \frac{1}{2}e^t,\ t \in (\ln 2, 2].$$

As in part (a) Mangasarian's Theorem implies that (x^*, u^*) is an optimal pair.

THE END