UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in	MAT2440 — Differential equations and optimal control theory
Day of examination:	11 June 2015
Examination hours:	0900 – 1300

This problem set consists of 6 pages.

Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 10 part questions will be weighted equally.

Problem 1 First order equations

1a

Solve the initial value problem

$$y' - 2xy = 4x^3 e^{x^2}, \qquad y(0) = 0.$$

Answer:

$$y' + P(x)y = Q(x),$$

where

$$P(x) = -2x, \quad Q(x) = 4x^3 e^{x^2}.$$

The integrating factor is

$$e^{\int P\,dx} = e^{-x^2}.$$

Multiplying the equation by this gives

$$(e^{-x^2} y)' = 4x^3,$$

and so

$$e^{-x^2} y = x^4 + C,$$

and

$$y = e^{x^2}(x^4 + C).$$

The initial condition implies $C = 0$, and so

$$y = e^{x^2} x^4.$$

(Continued on page 2.)

1b

Solve the equation

$$2xyy' = x^2 + 2y^2$$

by making the substitution $v = y/x$, or otherwise.

Answer:

$$2y' = (x/y) + 2(y/x).$$

Let $v = y/x$. Then $y = xv$ and $y' = xv' + v$, and so

$$2(xv' + v) = 1/v + 2v,$$

and so

$$2xv' = 1/v,$$

and so

$$2v\,dv = \frac{dx}{x},$$

and so

$$v^2 = \ln x + C,$$

and so

$$y^2 = x^2(\ln x + C).$$

Problem 2 Linear systems

Using the eigenvalue method, find the particular solution to the linear system $\boldsymbol{x}'(t) = A\boldsymbol{x}(t)$, with $\boldsymbol{x}(0) = \boldsymbol{x}_0$, where

$$A = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix}, \qquad \boldsymbol{x}_0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

Answer:

$$|A - \lambda I| = (2 - \lambda)(-2 - \lambda) + 20 = \lambda^2 + 16,$$

and so $\lambda = \pm 4i$. It is sufficient to use $\lambda = 4i$. To obtain an eigenvector we solve

$$(A - 4iI)\boldsymbol{v} = 0,$$

i.e.,

$$A = \begin{bmatrix} 2 - 4i & -5 \\ 4 & -2 - 4i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0,$$

and both equations are the same, and we can solve the first one by setting $a = 5$ and $b = 2 - 4i$, and then the eigenvector is $\boldsymbol{v} = [5, 2 - 4i]^T$. Then a (complex) solution to the equation is

$$\boldsymbol{x}(t) = \boldsymbol{v}e^{4it} = \left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ -4 \end{bmatrix} \right) (\cos 4t + i \sin 4t) = \boldsymbol{x}_1(t) + i\boldsymbol{x}_2(t),$$

(Continued on page 3.)

where

$$\boldsymbol{x}_1(t) = \begin{bmatrix}5\\2\end{bmatrix}\cos 4t - \begin{bmatrix}0\\-4\end{bmatrix}\sin 4t, \qquad \boldsymbol{x}_2(t) = \begin{bmatrix}5\\2\end{bmatrix}\sin 4t + \begin{bmatrix}0\\-4\end{bmatrix}\cos 4t,$$

and the general solution to the equation is

$$\boldsymbol{x}(t) = c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t).$$

For the particular solution, the initial condition gives

$$\begin{bmatrix}5\\6\end{bmatrix} = c_1\begin{bmatrix}5\\2\end{bmatrix} + c_2\begin{bmatrix}0\\-4\end{bmatrix},$$

whose solution is $c_1 = 1$, $c_2 = -1$, and hence,

$$\boldsymbol{x}(t) = \begin{bmatrix}5\cos 4t - 5\sin 4t\\6\cos 4t + 2\sin 4t.\end{bmatrix}$$

Problem 3 Matrix exponential

3a

Express the solution $\boldsymbol{x}(t)$, $t \geq 0$, to the initial value problem

$$\boldsymbol{x}'(t) = A\boldsymbol{x}(t) + \boldsymbol{f}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0,$$

in terms of the matrix exponential e^{tA}, where $\boldsymbol{x}(t)$, $\boldsymbol{f}(t)$, and $\boldsymbol{x}_0$ are vectors in $\mathbb{R}^n$, and A is a matrix in $\mathbb{R}^{n\times n}$.

Answer:

$$\boldsymbol{x}(t) = e^{tA}\boldsymbol{x}_0 + \int_0^t e^{(t-s)A}\boldsymbol{f}(s)\,ds.$$

3b

Find e^{tA} in the case that A is the matrix in Problem 2.
Answer: We use the fact that

$$e^{tA} = \Phi(t)\Phi(0)^{-1},$$

where $\Phi(t)$ is the fundamental matrix solution

$$\Phi(t) = \begin{bmatrix}\boldsymbol{x}_1(t) & \boldsymbol{x}_2(t)\end{bmatrix},$$

which is

$$\Phi(t) = \begin{bmatrix}5\cos 4t & 5\sin 4t,\\2\cos 4t + 4\sin 4t & 2\sin 4t - 4\cos 4t\end{bmatrix}.$$

Since

$$\Phi(0) = \begin{bmatrix}5 & 0\\2 & -4\end{bmatrix}, \qquad \Phi(0)^{-1} = -\frac{1}{20}\begin{bmatrix}-4 & 0\\-2 & 5\end{bmatrix},$$

we find

$$e^{tA} = -\frac{1}{4}\begin{bmatrix}-4\cos 4t - 2\sin 4t & 5\sin 4t\\-4\sin 4t & 2\sin 4t - 4\cos 4t\end{bmatrix}.$$

(Continued on page 4.)

Problem 4 Optimal control

Consider the problem

$$\max \int_0^1 (1-u(t))x(t)\,dt, \qquad \dot{x}(t) = 2u(t)x(t),$$

$$x(0) = 3, \qquad x(1) \text{ free},$$

with $u(t) \in U := [0,1]$, $t \in [0,1]$.

4a

If (x,u) is any admissible pair for this problem, explain why $x(t) > 0$ for all $t \in [0,1]$.

Answer: If not, since x is continuous, there is some t_* in $(0,1]$ such that $x(t_*) = 0$. Since $x(0) > 0$ we can further suppose that t_* is the smallest such t_*. Then by the differential equation and the control restriction $u \in U$, $\dot{x}(t) \geq 0$ for $t \in [0, t_*]$ and so $x(t_*) \geq x(0) = 3$, which is a contradiction.

4b

If the pair $(x^*(t), u^*(t))$ solves the problem, use the maximum principle to show that

$$u^*(t) = \begin{cases} 1, & t < 1/2; \\ 0, & t > 1/2. \end{cases}$$

Answer: The Hamiltonian is

$$H = (1-u)x + 2pux = x(1 + (2p-1)u),$$

with $p(t)$ the adjoint function. Therefore, due to the control restriction U and the fact that $x \geq 0$, if $p(t) > 1/2$, H is maximized by $u = u^* = 1$, and if $p(t) < 1/2$, H is maximimized by $u = u^* = 0$.

Since

$$\dot{p} = -\frac{\partial H^*}{\partial x} = -(1 + (2p-1)u^*),$$

we see that if $p(t) > 1/2$,

$$\dot{p}(t) = \begin{cases} -2p(t), & p(t) > 1/2; \\ -1, & p(t) \leq 1/2. \end{cases} \tag{1}$$

Thus p is monotonically decreasing in $[0,1]$. Since $x(1)$ is free, the transversality condition for p is $p(1) = 0$. This shows that there is some $t_* \in [0,1)$ such that $p(t) < 1/2$ for $t > t_*$. We can solve for p in the subinterval $[t_*, 1]$. Since $\dot{p}(t) = -1$ and $p(1) = 0$, we have $p(t) = 1 - t$ in this subinterval, and since $p(t_*) = 1/2$, we find that $t_* = 1/2$.

(Continued on page 5.)

4c

Find x^*.

Answer: Since $u^*(t) = 1$ for $t < 1/2$, solving the equation $\dot{x} = 2ux = 2x$ in the subinterval $[0, 1/2]$ with the initial condition $x(0) = 3$ gives $x(t) = 3e^{2t}$. For x in the subinterval $[1/2, 1]$, we have the equation $\dot{x} = 2ux = 0$ and the initial condition $x(1/2) = 3e$ from the first subinterval, which gives $x(t) = 3e$ for $t \in [1/2, 1]$.

4d

Show that the candidate (x^*, u^*) for optimality is indeed optimal.

Answer: If H is concave in x and u we can use the Mangasarian sufficiency theorem to show that (x^*, u^*) is optimal. However, since H contains the product term xu, it is not concave. However, we can try the weaker condition that

$$\hat{H}(x, p(t), t) := \max_{u \in U} H(x, u, p(t), t)$$

is concave in x (the Arrow sufficiency theorem). Since

$$\hat{H}(x, p(t), t) = \begin{cases} -2p(t)x, & t < 1/2; \\ -x, & t \geq 1/2, \end{cases} \tag{2}$$

it is concave in x for all $t \in [0, 1]$, and so Arrow's theorem applies.

Problem 5 Dynamical systems

Consider the non-linear system

$$\frac{dx}{dt} = xy - 2, \qquad \frac{dy}{dt} = x - 2y.$$

Find the critical points, their types, and stabilities.

Answer: The solutions to

$$xy - 2 = x - 2y = 0,$$

are $(x, y) = (2, 1)$ and $(x, y) = (-2, -1)$. The Jabobian of (f, g), $f(x, y) = xy - 2$, $g(x, y) = x - 2y$, is

$$J(x, y) = \begin{bmatrix} y & x \\ 1 & -2 \end{bmatrix}.$$

Let

$$J = J(2, 1) = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}.$$

Then

$$|J - \lambda I| = (1 - \lambda)(-2 - \lambda) - 2 = \lambda^2 + \lambda - 4,$$

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and so $\lambda = (-1 \pm \sqrt{17})/2$. Since the eigenvalues are real, with one negative, one positive, $(2, 1)$ is a saddle point, which is unstable.

Let

$$J = J(-2, -1) = \begin{bmatrix} -1 & -2 \\ 1 & -2 \end{bmatrix}.$$

Then

$$|J - \lambda I| = (-1 - \lambda)(-2 - \lambda) + 2 = \lambda^2 + 3\lambda + 4,$$

and so $\lambda = (-3 \pm \sqrt{7}i)/2$. Since the eigenvalues are a conjugate pair, with negative real part, $(-2, -1)$ is a sprial sink, which is asympotically stable.

Good luck!