Exercise

We want to solve numerically the ordinary differential equation

$$y' = f(y)$$

with initial data $y(0) = y_0$ on the interval $[0, T]$. As usual we split the interval $[0, T]$ into N subintervals and we denote $h = \frac{T}{N}$ and $t_k = kh$.

We consider the following numerical scheme. Given y_k (which corresponds to the approximation of y at time t_k), we compute

$$\begin{cases} m_{k,1} = f(y_k) \\ m_{k,2} = f(y_k + \dfrac{h}{3}\, m_{k,1}) \\ m_{k,3} = f(y_k + \dfrac{2h}{3}\, m_{k,2}) \end{cases}$$

and set

$$y_{k+1} = y_k + \frac{h}{4}(m_{k,1} + 3m_{k,3}).$$

1) We assume that f and all its derivatives are bounded by some constant M and that the time step h is smaller than 1. Prove that the local error for the first step of this method is of fourth order, that is,

$$|y(h) - y_1| \leq C(M)h^4$$

for some function $C(M)$ which depends only on M (and not on h).

2) What do you except the order of the total error to be for this method?