Exercise

1) We consider the problem

$$\min_{x(t)} \int_{t_0}^{t^1} F(x, \dot{x})\, dt,$$

so that the function F does not depend explicitely on t. Show that, in this case, the Euler-Lagrange equation (which, in general, is a second order differential equation) can be reduced to the following first order differential equation

$$F(x, \dot{x}) - \dot{x}\frac{\partial F}{\partial \dot{x}}(x, \dot{x}) = C,$$

where C is an arbitrary constant.

2) Brachistone problem. A particle of mass m moves under gravity in the (x, y)-plane, starting from rest at the origin O, along a smooth wire to a point A, which is no higher than O. What shape must the smooth wire OA have, in order to minimise the time taken for the journey?

We orient the vertical axis downwards and denote by $y(x)$ the function describing the shape of the wire. We assume that $A = (1, 1)$. The initial speed is zero. We can show that the time $T(y)$ for the particle to reach A from 0 is given by

$$T(y) = \frac{1}{\sqrt{2g}} \int_0^1 y^{-\frac{1}{2}}(1 + \dot{y}^2)^{\frac{1}{2}}\, dx. \tag{1}$$

We want to find the function $y(x)$ such that $y(0) = 0$ and $y(1) = 1$ which minimizes $T(y)$.

a) Find a necessary condition for the minimizer in form of a first order differential equation.

b) Show that the solution is a cycloide, that is, a curve which, in a parametrized form, writes

$$x(\theta) = a(\theta - \sin(\theta))$$
$$y(\theta) = a(1 - \cos(\theta))$$

where a is a constant.