UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in:	MAT2440 — DIFFERENTIAL EQUATIONS AND OPTIMAL CONTROL THEORY
Day of examination:	14.06.2011
Examination hours:	09.00 – 13.00

This problem set consists of 4 pages.

Appendices:	Last page
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 (weight 30%)

We consider the system of ordinary differential equations given by

$$\begin{aligned} \frac{dx}{dt} &= 4y^3 - 4xy \\ \frac{dy}{dt} &= 2y^2 - 2x^3. \end{aligned} \tag{1}$$

1a

Find all the equilibrium points of the system. Choose one of them (do not take the origo) and compute the linearization of the system around this equilibrium point. From the linearised system, determine, if you can, the stability of the equilibrium (attraction or repulsion point)

1b

Solve the differential equation

$$\frac{dy}{dx} = \frac{2y^2 - 2x^3}{4y^3 - 4xy}$$

(Hint: Rewrite the equation as an exact differential form.)

(Continued on page 2.)

1c

Show that the solutions of (1) satisfy

$$(x(t) - y^2(t))^2 + \frac{1}{2}(x^2(t) - 1)^2 = C,$$

for some constant C. Prove that the equilibrium point you considered in a) cannot be an attraction point.

Problem 2 (weight 30%)

2a

Find the solution of

$$\dot{X} = AX$$

for

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

and with initial value $X(0) = [0, 1, 0]^t$.

Problem 3 (weight 40%)

We want to solve

$$\max \int_0^\pi x^2 - u^2 \, dt$$

for $x(0) = 1$, $x(\pi)$ free , $\dot{x} = u$ and $u \in [0, 1]$.

3a

State the maximum principle. Show that p is strictly decreasing.

(Continued on page 3.)

3b

Show that

$$u^*(t) = \begin{cases} 0 & \text{if } p(t) < 0 \\ \frac{p(t)}{2} & p(t) \in [0, 2] \\ 1 & p(t) > 2 \end{cases}$$

3c

We can find a contant $t_* \in [0, \pi)$ such that $p(t) \in [0, 2]$ for $t \in [t_*, \pi]$. Explain why.

Let $\bar{x} = x(\pi)$. Compute $p(t)$ and $x(t)$ in the interval $[t_*, \pi]$ and write the result in term of t and $\bar{x}$.

3d

Explain why we cannot have $t_* = 0$. Let us now set t_* as the largest time such that $p(t_*) = 2$. Show that $\bar{x}\sin(t_*) = 1$.

3e

Let $\bar{p} = p(0)$. For $t \in [0, t_*]$, justify why we have $p(t) > 2$. For $t \in [0, t_*]$, Compute $x(t)$ and $p(t)$ as functions of $\bar{p}$ and t. Show that t_* satisfies

$$(t_* + 1) = -\frac{1}{\tan(t_*)} \tag{2}$$

and express $\bar{x}$ and $\bar{p}$ as function of t_*. Plot the optimal control $u^*(t)$.

(An approximate value of the unique solution of (2) in the interval $[0, \pi]$ is $t_* = 2.89$.)

(Continued on page 4.)

Appendix

Given a matrix $A \in \mathbb{R}^{n\times n}$ with eigenvalue λ, a generalized eigenvector of rank r (with $r \in \{1, \dots, n\}$) is a vector which satisfies

$$(A - \lambda I)^r u = 0$$
$$(A - \lambda I)^{r-1} u \neq 0.$$

Let m be the dimension of the eigenvector space, k the order of multiplicity of the eigenvalue λ. Then, any generalized eigenvector u for the eigenvalue λ satisfies

$$(A - \lambda I)^{k-m+1} u = 0$$

A chain of generalized eigenvector $\{v_1, \dots, v_r\}$ of length r satisfy

$$(A - \lambda I) v_{i+1} = v_i$$

for $i = 1, \dots, r-1$ and $v_r \neq 0$.

If $\{v_1, \dots, v_r\}$ is a chain of rank r for A, then

$$X_j(t) = e^{\lambda t} \left(\sum_{i=1}^{j} \frac{t^{j-i}}{(j-i)!} v_i \right)$$

is a solution to

$$\dot{X} = AX,$$

for all $j = 1, \dots, r$.