MANDATORY ASSIGNMENT - MAT2440

(DEADLINE: 14.04.2011)

In order to pass, you have to solve 60% of the assignment.

You can use any computer program (like Matlab or the numpy package of Python) to do the linear algebra (solving linear systems, computing eigenvalue, proceed to other matrix decompositions ...). When you do so, document the commands you are using and explain with details how you use the results.

Exercise 1: Solve

$$\frac{4y^2 - 2x^2}{4xy^2 - x^3}dx + \frac{8y^2 - x^2}{4y^3 - x^2 y}dy = 0$$

a) as an exact equation;

b) as a homogeneous equation.

Exercise 2:

a) We consider the solution of $\dot{X} = AX$ where

$$A = \begin{pmatrix} 2 & -1 \\ 2 & 3 \end{pmatrix}.$$

Find the general solution. Find the solution for the initial value $X(0) = [1, 0]^t$ and write it in the form

$$X(t) = re^{\alpha t}(\sin(\beta t + \gamma)v + \cos(\beta t + \gamma)w)$$

where r and γ are arbitrary constants and α, β real numbers, $v = [v_1, v_2]^t$ and $w = [w_1, w_2]^t$ vectors that you have to determine.

b) We consider the matrix

$$M = \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}.$$

Since MM^t is symmetric, it can be diagonalised in an orthonormal basis and the eigenvalues are positive: It implies that there exists $a, b \in \mathbb{R}$, $Q \in \mathbb{R}^{2\times 2}$ such that $Q^t Q = I$ and

$$MM^t = Q^t \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} Q.$$

Let us define the vector $X_0(t)$ as

$$X_0(t) = \sin(\beta t)v + \cos(\beta t)w. \tag{1}$$

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We change basis and, in the basis given by the transformation Q, the vector $X_0(t)$ rewrites as $\bar{X}_0(t)$ where

$$\bar{X}_0(t) = QX_0(t).$$

We denote $\bar{x}(t)$ and $\bar{y}(t)$ the components of $\bar{X}_0(t)$ (that is, $\bar{X}_0(t) = [\bar{x}(t), \bar{y}(t)]^t$). Show that

$$\left(\frac{\bar{x}(t)}{a}\right)^2 + \left(\frac{\bar{y}(t)}{b}\right)^2 = 1 \tag{2}$$

Hint: Write $X_0(t) = M\begin{pmatrix}\sin(\beta t)\\ \cos(\beta t)\end{pmatrix}$.

c) Equation (2) shows that $X_0(t)$, as defined by (1), moves along an ellipse. Compute Q and determine the axes of the ellipse (The computation can be done numerically, give the results with 10^{-3} accuracy). Plot the solutions $X(t)$ of **a)** for different values of r and γ.

Exercise 3: Find the general solution of $\dot{X} = AX$ where

$$A = \begin{pmatrix} 3 & 6 & 1 & 0 \\ -12 & -9 & -9 & -10 \\ 8 & 6 & 11 & 10 \\ 3 & 1 & 1 & 5 \end{pmatrix}.$$

Exercise 4: We consider the system of ordinary differential equations

$$\frac{dx}{dt} = \varepsilon x + y - x(x^2 + y^2), \tag{3a}$$

$$\frac{dy}{dt} = -x + \varepsilon y - y(x^2 + y^2), \tag{3b}$$

for $\varepsilon \in \mathbb{R}$.

a) Linearize the system and find the solutions of the linearized system.

b) We change to polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$. Show that for any functions $x(t)$ and $y(t)$, we have

$$\dot{r} = \dot{x}\cos\theta + \dot{y}\sin\theta$$

$$\dot{\theta} = -\frac{1}{r}(\dot{x}\sin\theta - \dot{y}\cos\theta)$$

and that, when $x(t)$ and $y(t)$ are solutions of (3), we have

$$\dot{r} = r(\varepsilon - r^2) \tag{4a}$$

$$\dot{\theta} = -1. \tag{4b}$$

What are the equilibrium points of (3)?

c) We assume $\varepsilon \leq 0$ and denote $\alpha = \sqrt{-\varepsilon}$. Solve (4). Show that $\lim_{t\to\infty} r(t) = 0$

d) We assume $\varepsilon \geq 0$ and denote $\alpha = \sqrt{\varepsilon}$. Solve (4). Show that $\lim_{t\to\infty} r(t) = \alpha$.