Ex 1.1.12

$$x^2 y'' - x y' + 2y = 0$$

* $y_1(x) = x\cos(\ln x)$

$$y_1'(x) = \cos(\ln x) + x(-\sin(\ln x))\frac{1}{x}$$

$$= \cos(\ln x) - \sin(\ln x)$$

$$y_1''(x) = \left[-\sin(\ln x) - \cos(\ln(x))\right]\frac{1}{x}$$

Hence,

$$x^2 y_1'' - x y_1' + 2y_1 = x^2 \cdot \frac{1}{x}\left[-\sin(\ln x) - \cos(\ln x)\right]$$
$$- x\left(\cos(\ln x) - \sin(\ln x)\right]$$
$$+ 2x\cos(\ln x)$$

$$= 0$$

and therefore y_1 is a solution.

* $y_2(x) = x\sin(\ln x)$

$$y_2'(x) = \sin(\ln x) + x\cos(\ln x)\frac{1}{x}$$

$$= \sin(\ln x) + \cos(\ln x)$$

$$y_2''(x) = \left[\cos(\ln x) - \sin(\ln x)\right]\frac{1}{x}$$

Hence, $x^2 y_2'' - x y_2' + 2y_2 = \frac{1}{x^2} \cdot \frac{1}{x}\left(\cos(\ln x) - \sin(\ln x)\right)$
$$- x\left(\sin(\ln x) + \cos(\ln x)\right)$$
$$+ 2x\sin(\ln x) = 0$$

and x is a solution.

Ex. 1.2.17

$$a(t) = \frac{1}{(t+1)^3}$$

$$\frac{dv}{dt} = a(t) = \frac{1}{(t+1)^3}$$

We integrate

$$v(t) = -\frac{1}{2}(1+t)^{-2} + C$$

$$v(0) = 0 \implies 0 = -\frac{1}{2} + C$$

$$\iff C = \frac{1}{2}$$

Hence, $v(t) = -\frac{1}{2}(1+t)^{-2} + \frac{1}{2}$

We have $\frac{dx}{dt} = v(t) = -\frac{1}{2}(1+t)^{-2} + \frac{1}{2}$

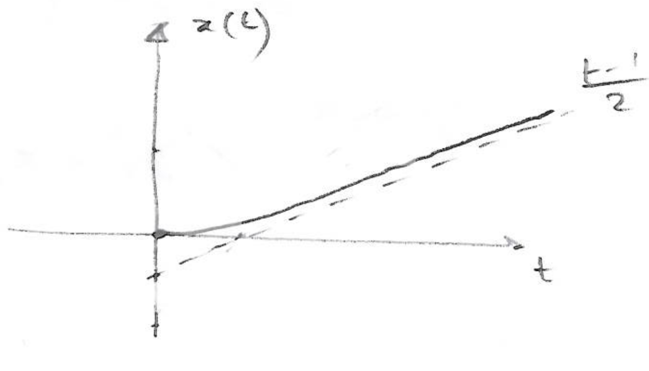
Hence,

$$x(t) = \frac{1}{2}(1+t)^{-1} + \frac{t}{2} + C$$

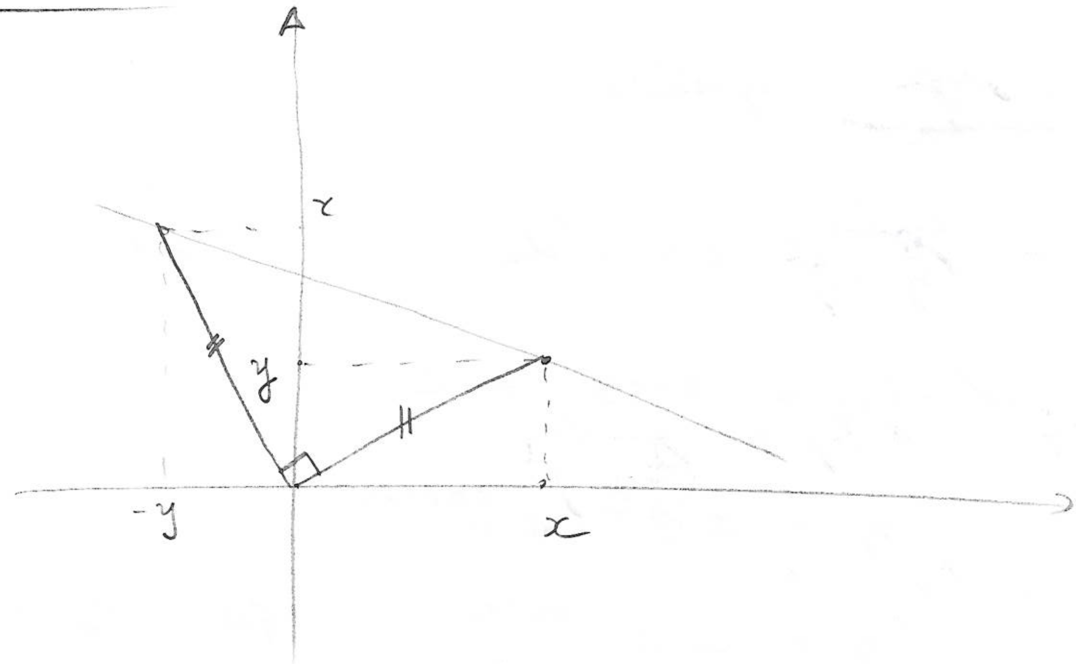
$$x(0) = 0 \implies \frac{1}{2} + C = 0$$

$$\iff C = -\frac{1}{2}$$

$$x(t) = \frac{1}{2(1+t)} + \frac{t}{2} - \frac{1}{2}$$



Ex 1.1.31



We compute the slope of the line which goes through (x, y) and and $(-y, x)$. Directional vector

$$\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} x + y \\ y - x \end{bmatrix}$$

Hence, the slope is

$$\frac{y - x}{y + x}$$

The equation is

$$\boxed{\frac{dy}{dx} = \frac{y - x}{y + x}}$$

Ex 1.4.3

$$\frac{dy}{dx} = y \sin x$$

$$\frac{dy}{y} = \sin x \, dx \qquad (y \neq 0)$$

$$\ln|y| = \cos(x) + c$$

$$y = \pm e^{c} e^{\cos(x)}$$

$$y = C e^{\cos(x)} \qquad \text{for } C \neq 0$$

We have $y(x) = 0$ is solution.
Hence, the general solution is

$$\boxed{y(x) = C e^{\cos x} \quad \text{for } C \in \mathbb{R}}$$

Ex 1.4.15.

$$\frac{dy}{dx} = \frac{(x-1)\, y^5}{x^2 (2y^3 - y)}$$

$$\left(\frac{2y^3 - y}{y^5}\right) dy = \left(\frac{x-1}{x^2}\right) dx$$

$$\int \frac{2y^3 - y}{y^5}\, dy = 2\int \frac{dy}{y^2} - \int \frac{1}{y^4}\, dy + c$$

$$= -\frac{2}{y} + \frac{1}{3} y^{-3} + c$$

$$\int \frac{x-1}{x^2}\, dx = \int \frac{1}{x}\, dx - \int \frac{1}{x^2}\, dx = \ln|x| - \frac{1}{x} + c$$

Hence,

$$-\frac{2}{y} + \frac{1}{3} y^{-3} = \ln|x| - \frac{1}{x} + C$$

The solution is only given implicitly because it is difficult to find an analytical expression for the inverse of $F : y \longmapsto -\frac{2}{y} + \frac{1}{3} y^{-3}$.

1.4.38

$F(t)$: value of the fund (in cents)

$\frac{dF}{dt} = \alpha F$ $\qquad \alpha = 5\%$

t : time in years

$F(t) = F(0) e^{\alpha t} = F_0 e^{\alpha t}$ $\qquad F_0 = 30$

$F(100) = 44{,}52$ \$

1.4.52

$N(t)$: number of language family

7 language family $\xrightarrow{\text{after 6000 years}}$ 1.5 language family

$$N(t) = N_0 (1{,}5)^t$$

This is exponential growth

$N(t) = N_0 e^{\ln(1.5)\, t}$ $\qquad \ln(1{,}5) = 0{,}4$

$$\frac{dN}{dt} = N_0 \ln(1,5)\, e^{\ln(1,5) t}$$

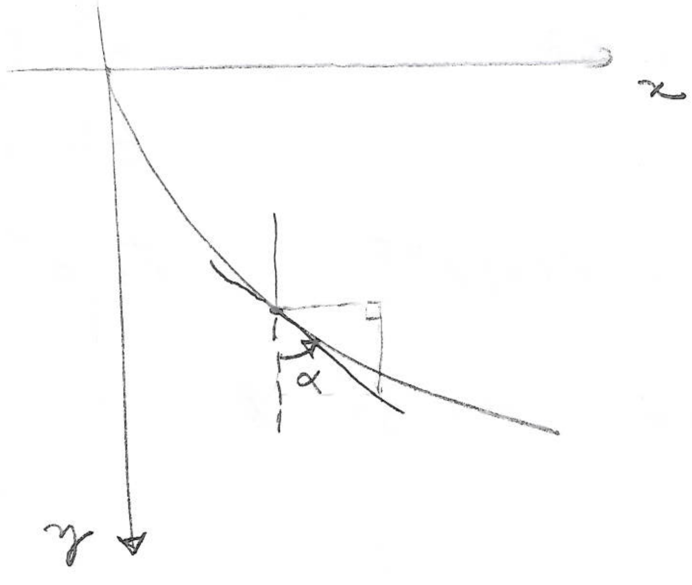
$$\boxed{\frac{dN}{dt} = \ln(1.5)\, N}$$

We have $\quad 3300 = 1.\,(1.5)^t \quad$ (t in 6000 years)

$$\Longrightarrow \quad t = \frac{\ln(3300)}{\ln(1,5)} \sim 19,98$$

$$= 120000 \text{ years}$$

Ex 1.4.68



$$y'(x) = \tan\left(\frac{\pi}{2} - \alpha\right) = \cot \alpha$$

$$\underbrace{\text{Kinetic Energy}}_{= \frac{1}{2} m v^2} + \underbrace{\text{potential energy}}_{= -mgy} = \underbrace{\text{constant in time}}_{= C}$$

The constant C is equal to zero as at $t=0$, $v(0)=0$ and $y(0)=0$

Hence, $\quad v = \sqrt{2gy}$

$$\frac{\sin\alpha}{v} = \text{constant} = b. \iff \sin\alpha = bv$$
$$= b\sqrt{2gy}$$

$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha} = \frac{\sqrt{1-\sin^2\alpha}}{\sin\alpha}$$

if $\alpha \in [0, \frac{\pi}{2}]$

$$= \frac{\sqrt{1 - b^2 2gy}}{b\sqrt{2gy}}$$

$$= \sqrt{\frac{2a-y}{y}} \qquad \text{for } a = \frac{1}{4b^2 g}$$

Hence

$$\frac{dy}{dx} = \sqrt{\frac{2a-y}{y}}$$

(b)

$$\frac{dy}{dx} = \sqrt{\frac{2a-y}{y}}$$

$$\sqrt{\frac{y}{2a-y}}\,dy = dx \qquad 2a - y \neq 0$$

$$\int_0^y \sqrt{\frac{y}{2a-y}}\,dy = \int_0^t \sqrt{\frac{2a\sin^2 t}{2a(1-\sin^2 t)}}\,4a\sin t \cos t\,dt$$

$$\left[\text{change of variable} \;\begin{cases} y = 2a\sin^2 t \\ dy = 4a\sin t\cos t\,dt \end{cases}\right]$$

$$= \int_0^t \frac{\sin t}{\cos t}\, 4a \sin t \cos t\,dt$$

$$= 4a\int_0^t \sin^2 t\,dt$$

$$= 4a \int_0^t \left[\frac{1-\cos(2t)}{2}\right] dt$$

$$= 4a\left(\frac{t}{2} - \frac{1}{4}\sin(2t)\right]$$

$$= a\left[2t - \sin(2t)\right]$$

Hence,

$$x = \int_0^x \sqrt{\frac{y}{2a-y}}\,\frac{dy}{dx}\,dx = \int_0^{y(x)} \sqrt{\frac{\tilde{y}}{2a-\tilde{y}}}\,d\tilde{y}$$

Change of variable (separation of variable method)

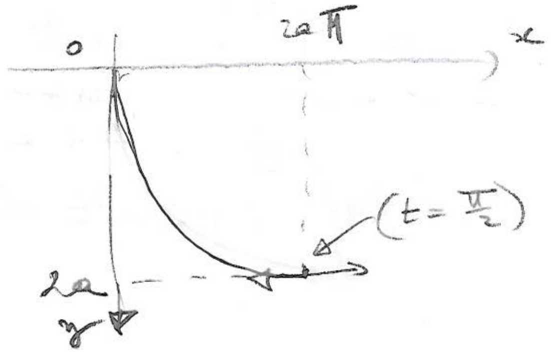
change of variable $\tilde{y} = 2a\sin^2\tilde{t}$ ($\Rightarrow$ t is such that $y(x) = 2a\sin^2 t = a(1-\cos 2t)$)

$$= 4a\int_0^t \sin^2 t\,dt$$

$$= a(2t - \sin 2t)$$

Hence, t is such that

$$\begin{cases} x(t) = a(2t - \sin 2t) \\ y(t) = a(1-\cos 2t) \end{cases}$$



$$\dot{x}(t) = 2a - 2a\cos 2t = 2a(1-\cos 2t)$$

t	0		π
$\dot{x}(t)$	0	$+$	0
$x(t)$		$\nearrow$	

for $t \in (0,\pi)$, $x(t)$ is strictly increasing

$\Rightarrow$ x is invertible

$\Rightarrow$ There exists function

$$\bar{x} \longmapsto t(\bar{x})$$

$$x(t(\bar{x})) = \bar{x} \qquad \text{for all } \bar{x}$$

$$\frac{d}{dx}\left(x(t(\bar{x}))\right) = 1$$

$$\Leftrightarrow \quad \dot{x}(t(\bar{x}))\, \frac{dt}{dx}(\bar{x}) = 1$$

$$\Rightarrow \quad \frac{dt}{dx}(\bar{x}) = \frac{1}{\dot{x}(t(\bar{x}))}$$

Hence,

$$y(\bar{x}) = a\left(1 - \cos[2t(\bar{x})]\right) = 2a\sin^2[t(\bar{x})]$$

$$\frac{dy}{d\bar{x}} = 2a\frac{dt}{d\bar{x}}\sin[2t(\bar{x})]$$

$$= 2a\frac{\sin(2t(\bar{x}))}{2a - 2a\cos[2t(\bar{x})]}$$

$$= \frac{\sin[2t(\bar{x})]}{1-\cos[2t(\bar{x})]} = \frac{2\sin(t(\bar{x}))\cos(t(\bar{x}))}{2\sin^2(t(\bar{x}))}$$

$$\frac{dy}{d\bar{x}} = \frac{\cos t(\bar{x})}{\sin t(\bar{x})}$$

$$= \sqrt{\frac{2a - y}{y}}$$