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a)

$$A(0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0 \iff \lambda = \pm i$$

Let us consider $\lambda = i$

$$[A - \lambda I]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \iff [-i \quad -1]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

We take $u = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

We obtain the complex-valued solution

$$x(t) = e^{it}\begin{bmatrix} 1 \\ -i \end{bmatrix} = (\cos t + i \sin t)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + i \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$$

the general real-valued solution is

$$x(t) = c_1 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$$

b)

$$E(t) = x(t)^2 + y(t)^2$$

$$\frac{dE}{dt} = 2\dot{x}x + 2\dot{y}y = -2yx + 2(x + ay)y$$

$$= 2ay^2 \leq 0$$

because $a \leq 0$. Therefore,

E is decreasing.

c)

$$|A - \lambda I| = 0 \iff \lambda^2 - a\lambda + 1 = 0$$

If the roots are complex, that is,

$$a^2 - 4 \leqslant 0$$

then $Re(\lambda) = a$ and $Re(\lambda) < 0 \iff a < 0$

If the roots are real, we have

$$\lambda_1 + \lambda_2 = a \quad (1)$$

and $\lambda_1 \lambda_2 = 1 \quad (2)$

and $\begin{cases} \lambda_1 < 0 \\ \lambda_2 < 0 \end{cases} \iff a < 0$

Indeed, if $a < 0$, then, as the product is strictly positive, the eigenvalues must have the same sign and none of them can be equal to zero. From (1), this sign can only be negative.

Conclusion: Both eigenvalues are strictly negative if and only if $a < 0$.

Complex valued solution if $\lambda_1 \neq \lambda_2$

$$X(t) = c_1 e^{\lambda_1 t} u_1 + c_2 e^{\lambda_2 t} u_2$$

Since $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$, we have

$$\lim_{t \to \infty} X(t) = 0$$

We can have $\lambda_1 = \lambda_2$ when $a^2 = 4$.
The eigenvalue is negative only when $a < 0$ that $a = -2$.
Then, $\lambda = -1$. We have

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}.$$

There are two possibilities:

* Either A is diagonalisable and the conclusion is obtained as in the case $\lambda_1 \neq \lambda_2$

* Either A is not diagonalisable. We introduce the chain of generalized eigenvector $\{v_1, v_2\}$. The general solution is given by

$$X(t) = C_1 e^{-2t} v_1 + C_2 e^{-2t}(t v_1 + v_2)$$

Since $\lim_{t \to \infty} t e^{-2t} = 0$, we get

$$\lim_{t \to \infty} X(t) = 0$$ also in this case.

d) Gronwalls lemma: If $y : \mathbb{R} \to \mathbb{R}$ satisfies $\dot{y} \leq \alpha y$ for some positive constant α, then

$$y(t) \leq y(0) e^{\alpha t}$$

We compute earlier that

$$\frac{dE}{dt} = 2ay^2$$

$$\leq 2\sin(t^2)\, y^2$$

$$\leq 2y^2$$

$$\leq 2E \qquad (E = x^2 + y^2)$$

By Gronwall's lemma

$$E(t) \leq E(0)\, e^{2t}$$

$$\leq e^{2t} \qquad (E(0) = 1^2 + 0^2 = 1)$$

Hence

$$|x(t)| \leq \sqrt{x^2 + y^2} = \sqrt{E} \leq e^t$$

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Exercise 2

a)

$$H = (tx - u^2) + p(x + u^2)$$

$$H = (t+p)x + (p-1)u^2$$

$$\frac{\partial H}{\partial u} = 0 \iff u = 0$$

The maximum is either reached at $u=0$ or at the boundary of the control region ($u=0$, $u=1$)

We have $H(1) - H(0) = (p-1)$

and $u^* = 0$ if $p-1<0$

$u^* = 1$ if $p-1>0$

We have

$$\dot{p} = -\frac{\partial H}{\partial x} = -(t+p)$$

$$\dot{p} + p = -t$$

A particular solution is $-t+1$.

The general solution is

$$p(t) = Ae^{-t} + 1 - t$$

We have $p(1) = 0$ so that $A = 0$

and $p(t) = 1 - t$

Then $\begin{cases} p(t) > 1 & \text{for } t < 0 \\ p(t) < 1 & \text{for } t > 0 \end{cases}$

For $t < 0$, $p > 1 \Rightarrow u^* = 1$ and

$$\dot{x} = x + u^{*2} = x + 1$$

A particular solution is -1 and the general solution is

$$x(t) = Ae^{t} - 1$$

$x(-1) = -2e^{-1} - 1 \Rightarrow -2e^{-1} - 1 = Ae^{-1} - 1$

$$\Rightarrow A = -2$$

$$\boxed{x(t) = -2e^{t} - 1} \quad (*) \quad \text{for } t < 0$$

for $t > 0$, we have $p < 1$ and $u^* = 0$

$$\dot{x} = x$$

$$x(t) = Ae^{t}$$

$x(0) = -2 - 1 = A \Rightarrow A = -3$

from $(*)$

and $\boxed{x(t) = -3e^{t} \quad \text{for } t > 0}$

b) This question requires Arrow's condition, which we did not study in the course.

Arrow's ~~su~~ sufficient condition (SSS p. 385):

$$x \mapsto \hat{H}(t, x, p(t)) \text{ is concave for all } t \in [-1, 1]$$

where $\hat{H}(t, x, p) = \max_{u \in [0,1]} H(t, x, u, p)$

We have in this case

$$\hat{H}(t, x, p) = \begin{cases} (1 + p(t))x + (p(t) - 1) & \text{for } t \in [-1, 0] \\ (1 + p(t))x & \text{for } t \in [0, 1] \end{cases}$$

Hence, the condition is satisfied.

Note that the function

$$(x, u) \mapsto H$$

is not concave for $t \in [0, 1]$ and we really need to use "Arrow's sufficient condition.

c) We have $p(t) = Ae^{-t} + 1 - t$, as before.

As indicated in the text, we look for a solution such that

$$p(t) < 1 \quad \text{in } (t_0, 1]$$

for some $t_0 \in [-1, 1)$.

Then, ~~for~~ $t \in (t_0, 1]$, we have $u^* = 0$ and $\dot{x} = x$

which gives

$$x(t) = Be^{t}.$$

$$x(1) = e^{2} - e^{1+\frac{1}{e}} \implies Be = e^{2} - e^{1+\frac{1}{e}}$$

$$B = e - e^{\frac{1}{e}}$$

and $\quad x(t) = e^{1+t} - e^{\frac{1}{e}+t}.$

We cannot have $t_0 = -1$ because, then, $\quad x(-1) = 1 - e^{\frac{1}{e}-1} \neq 0$

We assume that there exists $t_0 \in [-1, 1)$ such that

$$p(t_0) = 1$$

It implies $\quad 1 = Ae^{-t_0} + 1 - t_0$

$$\implies A = t_0 e^{t_0}$$

and $\quad p(t) = t_0 e^{t_0 - t} + 1 - t$

For $t < t_0$, we have $p(t) > 1$ because p is decreasing. Then, $u^* = 1$

and $\quad \dot{x} = x + 1$

which gives $\quad x(t) = Ce^{t} - 1$

$$x(-1) = 0 \implies Ce^{-1} - 1 = 0$$

$$\implies C = e$$

and $\quad x(t) = e^{1+t} - 1$

By the continuity of $x(t)$ at t_0 we get

$$e^{1+t_0} - 1 = e^{1+t_0} - e^{\frac{1}{e} + t_0}$$

$$\Longleftrightarrow \quad -1 = -e^{\frac{1}{e} + t_0}$$

$$\Longleftrightarrow \quad \frac{1}{e} + t_0 = 0$$

$$\Longleftrightarrow \quad t_0 = -\frac{1}{e}$$

We have $t_0 \in [-1, 1)$.

The pair

$$u^*(t) = \begin{cases} 1 & \text{for } t \in [-1, \frac{-1}{e}] \\ 0 & \text{for } t \in [-\frac{1}{e}, 1] \end{cases}$$

$$x^*(t) = \begin{cases} e^{1+t} - 1 & \text{for } t \in [-1, \frac{-1}{e}] \\ e^{1+t} - e^{\frac{1}{e}+t} & \text{for } t \in (-\frac{1}{e}, 1] \end{cases}$$

is the solution.

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$$\max_u \int_0^1 (\ln u + x)\,dt \qquad \dot{x} = x - u$$

$$x(0) = 0$$

$$x(1) \text{ is free}$$

$u \in (0, 1]$

a)

$$H = \ln u + x + p(x-u)$$

$$\frac{\partial H}{\partial x} = 1 + p$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -(1+p)$$

$$\dot{p} + p = -1$$

A particular solution is -1.
The general solution is

$$p(t) = Ae^{-t} - 1$$

$x(1)$ is free $\Rightarrow$ $\cancel{x}p(1) = 0$

$$\Rightarrow Ae^{-1} - 1 = 0$$

$$\Rightarrow A = e$$

and $\boxed{p(t) = e^{1-t} - 1}$

b)

$$p(t_0) = 1 \iff e^{1-t_0} - 1 = 1$$

$$\iff \boxed{t_0 = 1 - \ln 2}$$

We have

$$H(u^*) = \max_{u \in (0,1]} H(u)$$

$$\frac{\partial H}{\partial u} = 0 \iff \frac{1}{u} - p = 0$$

$$\iff u^* = \frac{1}{p}$$

If $\frac{1}{p} \in (0,1]$ then the maximum of H is reached for $u^* = \frac{1}{p}$ because the function is concave.

$\frac{1}{p} \in (0,1]$ if $p > 1$.

Since p is decreasing, $p(t) > 1$ for $t < t_0$ and therefore

$$u^*(t) = \frac{1}{p(t)} \quad \text{for } t < t_0$$

c) $\dot{x} = x - u^*$

For $t \in [0, t_0)$, $u^* = \frac{1}{p(t)} = \frac{1}{e^{1-t} - 1}$

$$\dot{x} - x = -\frac{1}{e^{1-t} - 1}$$

Integrating factor μ: $\frac{\dot{\mu}}{\mu} = -1$

$$\Rightarrow \mu = e^{-t}$$

$$\dot{\overline{(\mu x)}} = \frac{-e^{-t}}{e^{1-t} - 1}$$

$$\int \frac{-e^{-t}}{e^{1-t}-1}\,dt \underset{\substack{\uparrow \\ v=e^{-t} \\ dv=-e^{-t}dt}}{\overset{}{=}} \int \frac{dv}{ev-1} = \frac{1}{e}\ln|ev-1|$$

Hence

$$e^{-t}x = \frac{1}{e}\ln|e^{1-t}-1| + A$$

$$x(t) = e^{t-1}\ln(e^{1-t}-1) + Ae^{t}$$

$$x(0)=0 \implies e^{-1}\ln(e-1)+A=0$$

$$x(t) = e^{t-1}\ln(e^{1-t}-1) - e^{t-1}\ln(e-1)$$

$$\boxed{x(t) = e^{t-1}\ln\left[\frac{e^{1-t}-1}{e-1}\right]}$$

d)

$$\frac{\partial H}{\partial u} = \frac{1}{u} - p\,.$$

When $t \in (t_0, 1]$, $p(t) < 1$ and

$$\frac{1}{u} - p > \frac{1}{u} - 1 \geqslant 0$$

because $u \in [0,1]$

Hence $\frac{\partial H}{\partial u} > 0$ for all $u \in (0,1]$.

The function H which is concave and such that $\lim_{u\to 0} H(u) = -\infty$ can only reach its maximum at $u=1$.

Hence,

$$u^* = 1 \quad \text{for} \quad t \in (t_0, 1].$$

e) For $t \in (t_0, 1]$, $u^* = 1$ implies

$$\dot{x} = x - 1 .$$

It gives
$$x(t) = Ae^{t} + 1$$
(particular solution : 1)

To determine A we use the continuity of $x(t)$ at $t = t_0$

$$x(t_0) = e^{t_0 - 1} \ln \left[\frac{e^{1-t_0} - 1}{e - 1} \right] = Ae^{t_0} + 1$$

Using the fact that $e^{1-t_0} = 2$, we get

$$\frac{1}{2} \ln \left[\frac{1}{e-1} \right] = A \frac{e}{2} + 1$$

$$A = -2e^{-1} + e^{-1} \ln \left[\frac{1}{e-1} \right]$$

$$A = -e^{-1} \left[2 + \ln(e-1) \right]$$

$$\boxed{x(t) = -e^{t-1} \left[2 + \ln(e-1) \right] + 1 \quad \text{for } t \in (t_0, 1].}$$

$(x, u) \longmapsto \ln u + x + p(x - u)$ is concave because $u \mapsto \ln u$, $u \longmapsto -pu$, $x \longmapsto x + px$ are concave. Therefore (x^*, u^*) which satisfies the maximum principle is a solution of the optimisation problem.