1.3.1 check with matlab/ vector field plotter

1.3.11

- The function $f(x,y)=x^2y^2$ is continuous on $\mathbb{R}^2$.
- $\frac{\partial f}{\partial y} = 2x^2y$ is continuous on $\mathbb{R}^2$

Hence, conditions for existence and uniqueness are fullfilled.

1.3.12

$$\frac{dy}{dx} = x \ln y \qquad y(1) = 1$$

The function $y \longmapsto \ln y$ is continuous on $(0, \infty)$

$\Longrightarrow$ $f(x,y) = x \ln y$ is continuous on $\mathbb{R} \times (0, \infty)$

On $\mathbb{R}\times(0,\infty)$, we have

$\frac{\partial}{\partial y} f = \frac{x}{y}$ which is continuous

on $\mathbb{R} \times (0,\infty)$ (we do not consider $\mathbb{R} \times (-\infty, 0)$ as $\frac{\partial f}{\partial y}$ is not defined there)

As far as the point $(1,1)$ is concerned, it belongs to region where f, $\frac{\partial f}{\partial y}$ are continuous and there exists a unique solution locally going through this point.

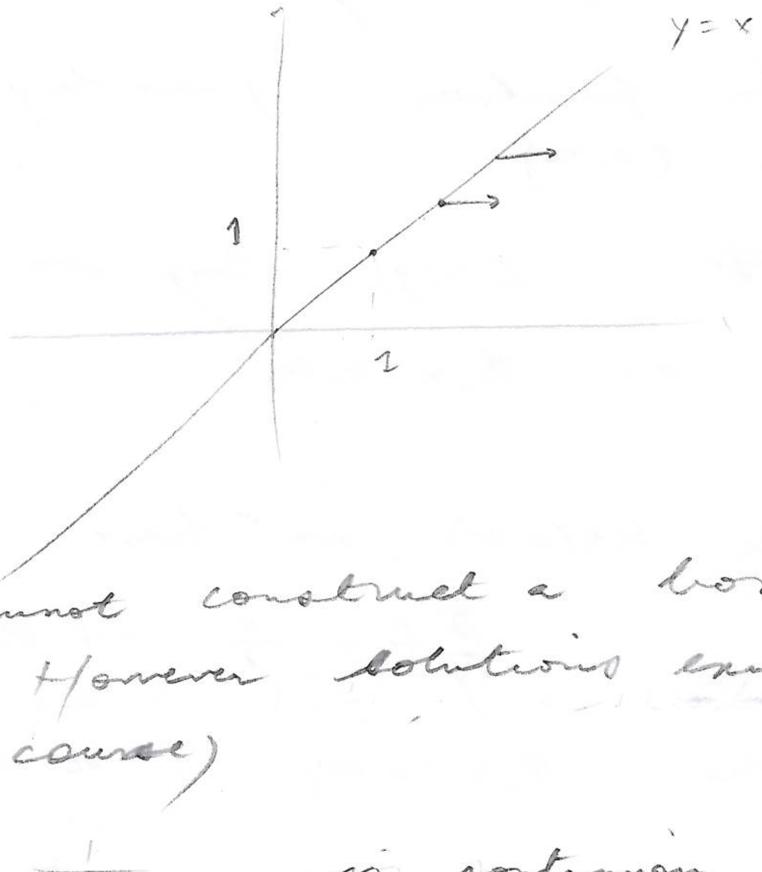
1.3.15

$$\frac{dy}{dx} = \sqrt{x-y}$$

- The function $z \to \sqrt{z}$ is continuous in $[0,\infty)$

- The function $x-y$ is continuous on $\mathbb{R}^2$

$K = \{ (x,y) \in \mathbb{R}^2 \mid x-y \geq 0 \}$



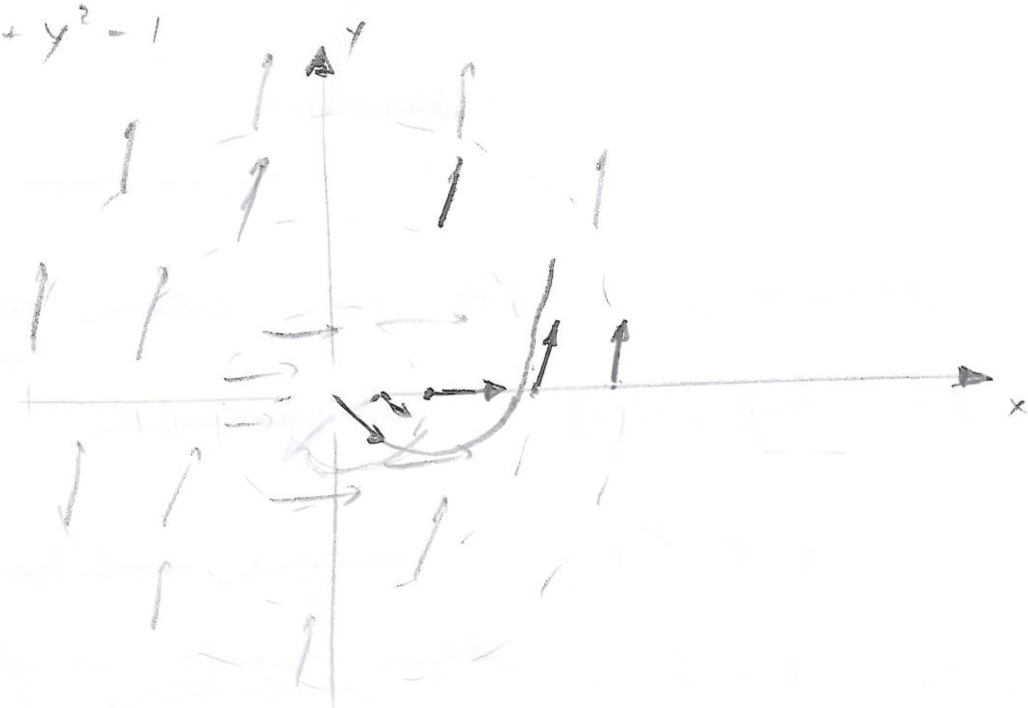
We cannot construct a box around point. However solutions exist (see course)

- $\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{x-y}}$ is continuous in $\mathring{K} = \{ (x,y) \in \mathbb{R}^2 \mid x-y > 0 \}$

The solution starting at $(1,1)$ is not unique.

1.3.23

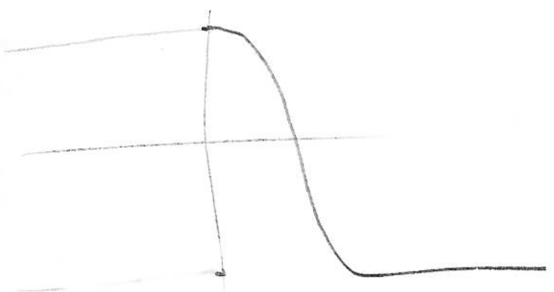
$y' = x^2 + y^2 - 1$



Plot for $z = 0$. Then use rotation symmetry.

1.3.30

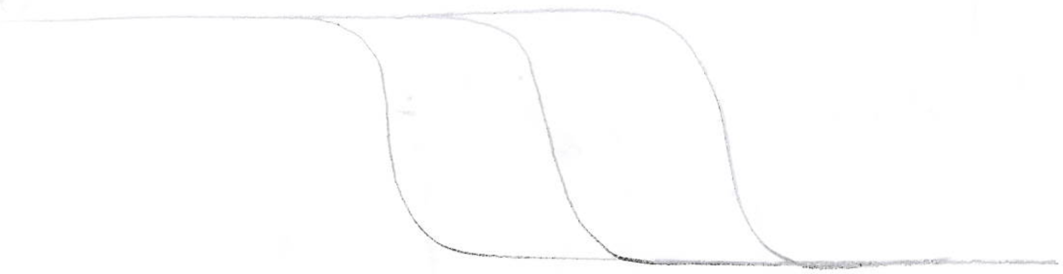
$$y(x) = \begin{cases} 1 & x \leq 0 \\ \cos(x) & 0 < x < \pi \\ -1 & x \geq \pi \end{cases}$$



* If $x \leq 0$

$$\frac{dy}{dx} = 0 = y(x)$$

* If $x \in (0, \pi)$ $\quad \frac{dy}{dx} = -\sin x = -\sqrt{1-y^2}$
* If $x \geq \pi$ $\quad \frac{dy}{dx} = 0$ (see end p. 3)



$y(a) = b$. We can choose a arbitrarily (autonomous equation)

If $b \notin [-1, 1]$, no solution

If $b \in (-1, 1)$, unique solution locally

If $b = 1$, there exists infinitely many solutions

If $b = -1$, there exists at least one solution.

1.11.3)

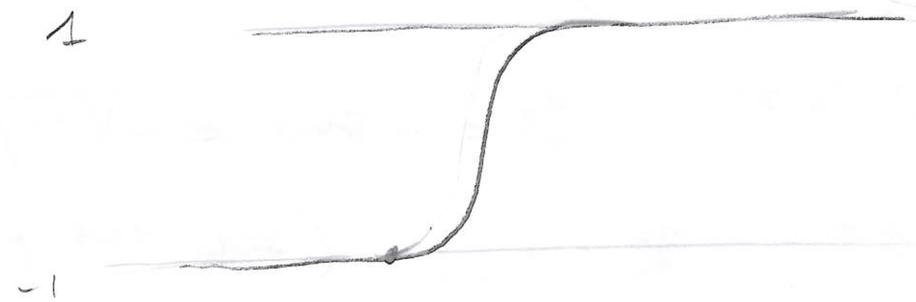
$$y' = \sqrt{1-y^2}$$

$$\frac{dy}{dx} = \sqrt{1-y^2}$$

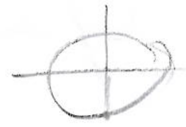
$$\frac{dy}{\sqrt{1-y^2}} = dx$$

$$\arcsin y = x - c \qquad (x-c) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

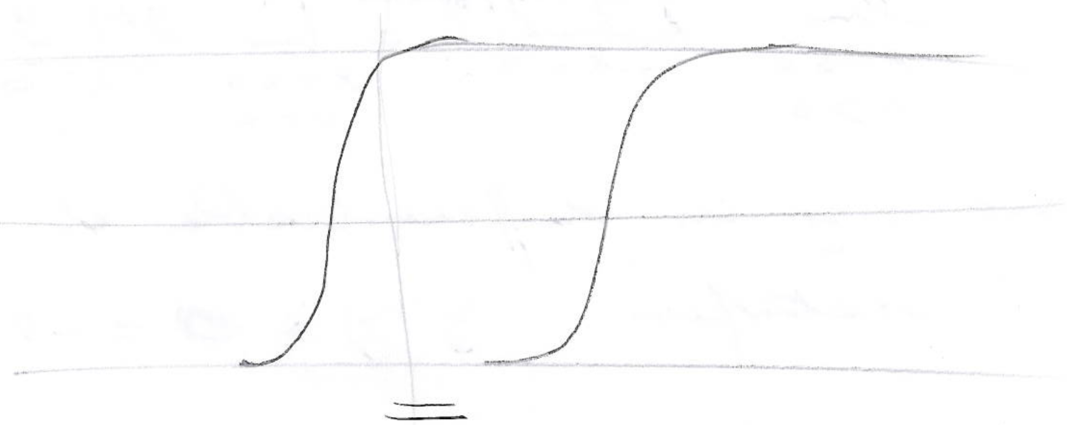
$$y = \sin(x-c)$$



$$y(x) = \begin{cases} -1 & x \leq -\frac{\pi}{2} \\ \sin x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 1 & x \geq \frac{\pi}{2} \end{cases}$$



$$y(x) = \begin{cases} -1 & \text{if } x < -\frac{\pi}{2} + c \\ \sin(x-c) & \text{if } -\frac{\pi}{2} + c < x < \frac{\pi}{2} + c \\ 1 & \text{if } x \geq \frac{\pi}{2} + c \end{cases}$$



Ex 1.3.30 (continued)

We have to prove that $y(x)$ is differentiable at $x = 0$ and $x = \pi$ and that it satisfies the ODE at those points.

We just check for $x = 0$.

- $$\lim_{\substack{x \to 0 \\ x < 0}} \frac{y(x) - y(0)}{x - 0} = 0$$

because $y(x) - y(0) = 0$ for $x < 0$

- For $x > 0$ (and x close enough to 0), we have

$$y(x) - y(0) = \cos(x) - 1 = \cos(x) - \cos(0)$$

Hence,

$$\lim_{\substack{x \to 0 \\ x > 0}} \frac{y(x) - y(0)}{x - 0} = \lim_{\substack{x \to 0 \\ x > 0}} \frac{\cos(x) - \cos(0)}{x - 0}$$

$$= -\sin(0) = 0$$

So that

$$\lim_{\substack{x \to 0 \\ x > 0}} \frac{y(x) - y(0)}{x - 0} = \lim_{\substack{x \to 0 \\ x < 0}} \frac{y(x) - y(0)}{x - 0} = 0$$

and y is differentiable at zero and satisfies $y'(0) = 0 = -\sqrt{1 - y^2(0)}$.