1.5.24 :

$$(x^2+4)y' + 3xy = x \qquad y(0)=1$$

$$y' + \frac{3x}{x^2+4}y = \frac{x}{x^2+4}$$

linear equation.

Integrating factor:

$$\mu(t) = e^{\int \frac{3x}{x^2+4}dx}$$

$$= e^{\frac{3}{2}\ln(x^2+4)}$$

$$= (x^2+4)^{3/2}$$

$$\frac{d}{dt}(\mu y) = \mu \frac{x}{x^2+4} = x(x^2+4)^{\frac{1}{2}}$$

$$\mu y = \frac{1}{3}(x^2+4)^{3/2} + C \qquad (C \in \mathbb{R} \text{ constant})$$

$$(x^2+4)^{3/2}y = \frac{1}{3}(x^2+4)^{3/2} + C$$

$$\boxed{y(x) = \frac{1}{3} + \frac{C}{(x^2+4)^{3/2}}}$$

$$y(0)=1 \iff \frac{1}{3} + \frac{C}{4^{3/2}} = 1$$

$$\iff \frac{C}{8} = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\boxed{y(x) = \frac{1}{3} + \frac{16}{3(x^2+4)^{3/2}}}$$

1.5.25

$$\frac{dy}{dx} + \frac{3x^3}{x^2+1} y = \frac{6x e^{-3/2 x^2}}{x^2+1} \qquad y(0)=1$$

$$\mu = e^{\int \frac{3x^3}{x^2+1} dx}$$

$$\int \frac{3x^3}{x^2+1} dx = \int \frac{3x(x^2+1) - 3x}{x^2+1} dx$$

$$= \frac{3}{2} x^2 - \frac{3}{2} \ln(x^2+1)$$

$$= \frac{3}{2} \left(x^2 - \ln(x^2+1)\right)$$

$$\frac{d}{dx}(\mu y) = e^{3/2(x^2 - \ln(x^2+1))} \frac{6x}{x^2+1} e^{-3/2 x^2}$$

$$= \frac{6x}{x^2+1} (x^2+1)^{-3/2}$$

$$= 6x (x^2+1)^{-5/2}$$

$$\mu y = -4(x^2+1)^{-3/2} + C$$

$$y(x) = e^{-3/2 x^2}(x^2+1)^{3/2}\left(-4(x^2+1)^{-3/2}\right) + C e^{-\frac{3}{2}x^2}(x^2+1)^{3/2}$$

$$\boxed{y(x) = -4e^{-3/2 x^2} + C e^{-\frac{3}{2}x^2}(x^2+1)^{3/2}}$$

$$y(0) = 1 \implies 1 = -4 + C$$

$$\implies C = 5$$

Hence,

$$\boxed{y(x) = -4e^{-\frac{3}{2}x^2} + 5e^{-\frac{3}{2}x^2}(x^2+1)^{3/2}}$$

1.6.5

$$x(x+y)y' = y(x-y)$$

$$1\left(1+\frac{y}{x}\right)y' = \frac{y}{x}\left(1-\frac{y}{x}\right) \quad \text{after dividing both sides by } x$$

$$y' = F(v)$$

for $F(v) = \dfrac{v(1-v)}{1+v}$ and $v = \dfrac{y}{x}$

We make the substitution $v = \dfrac{y}{x}$

$$vx = y$$

$$\frac{dv}{dx}x + v = \frac{dy}{dx} = \frac{v(1-v)}{1+v}$$

$$x\frac{dv}{dx} = \frac{v(1-v)}{1+v} - v$$

$$= \frac{v - v^2 - v^2 - v}{1+v}$$

$$= -\frac{2v^2}{1+v}$$

$$\frac{v+1}{v^2}\,dv = -\frac{2}{x}\,dx \qquad (\text{for } v \neq 0)$$

$$\left(\frac{1}{v} + \frac{1}{v^2}\right) dv = -\frac{2}{x}\,dx$$

$$\left(\ln|v| - \frac{1}{v}\right) = -2\ln|x| + c$$

$$|v|e^{-\frac{1}{v}} = e^{c}\,\frac{1}{|x|^2}$$

$$ve^{-\frac{1}{v}} = \frac{C}{x^2} \qquad \text{for } C \in \mathbb{R}\setminus\{0\}$$

We can take $C=0$ and recover the solution $v(x) = 0$.

$$\frac{y}{x}e^{-\frac{y}{x}} = \frac{C}{x^2}$$

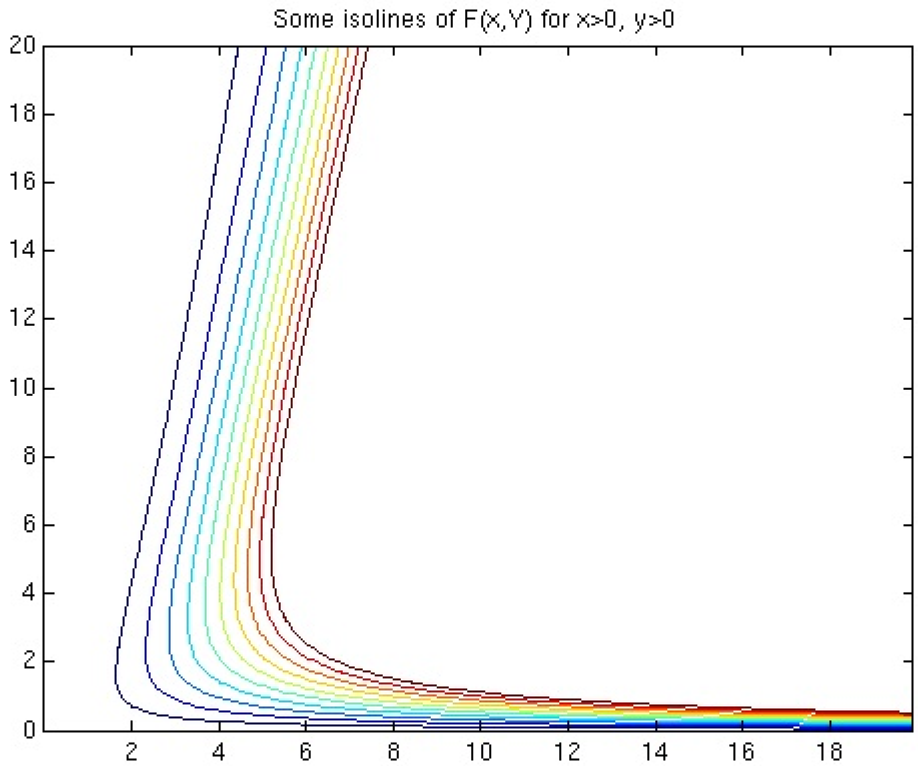
$$ye^{-\frac{y}{x}} = \frac{C}{x}$$

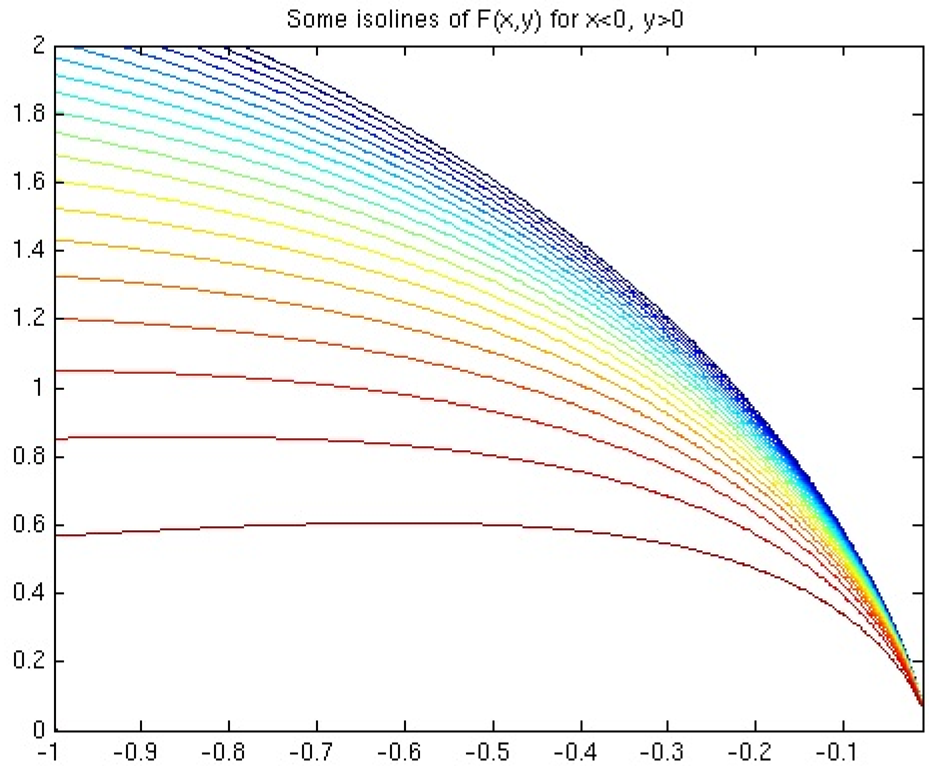
$$\boxed{xye^{-\frac{y}{x}} = C}$$

$$F(x,y) = C$$

where $F(x,y) = xye^{-\frac{y}{x}}$.

```
function y=main()

  % for x<0 and y>0
  % [X,Y]=meshgrid(-1:0.005:-0.01,0:0.005:2);

  % for x>0 and y>0
  [X,Y]=meshgrid(0.01:0.1:20,0:0.1:20);

  Z=X.*Y.*exp(-Y./X);

  % for x<0 and y>0
  % v=[-20:1:0]

  % for x>0 and y>0
  v=[1:10]

  contour(X,Y,Z,v)

end
```


Some isolines of F(x,Y) for x>0, y>0
20
18
16
14
12
10
8
6
4
2
0
2
4
6
8
10
12
14
16
18


Some isolines of F(x,y) for x<0, y>0
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-1
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1

1.6.11

$$(x^2-y^2)\,y' = 2xy$$

$$\left(1-\frac{y^2}{x^2}\right) y' = 2\,\frac{y}{x}$$

$$y' = F(v)$$

for $v = \frac{y}{x}$ and $F(v) = \frac{2v}{1-v^2}$

Then, v satisfies

$$x\,\frac{dv}{dx} = F(v) - v$$

$$= \frac{2v}{1-v^2} - v$$

$$= \frac{v+v^3}{1-v^2}$$

$$x\,\frac{dv}{dx} = \frac{v(1+v^2)}{1-v^2}$$

$$\frac{(1-v^2)}{v(1+v^2)}\,dv = \frac{dx}{x}$$

$$\int \frac{1-v^2}{v(1+v^2)}\,dv = \int \frac{v(1-v^2)}{v^2(1+v^2)}\,dv$$

$$\begin{bmatrix} u = v^2 \\ du = 2v\,dv \end{bmatrix}$$

$$= \frac{1}{2}\int \frac{1-u}{u(1+u)}\,du$$

$$\frac{1-u}{u(1+u)} = \frac{1}{u} + \frac{-2}{1+u}$$

Hence

$$\frac{1}{2}\int \frac{1-u}{u(1+u)}\,du = \frac{1}{2}\left[\ln u - 2\ln(1+u)\right]$$

$$= \ln \frac{\sqrt{u}}{1+u}$$

$$= \ln\left[\frac{|v|}{1+v^2}\right]$$

Finally

$$\ln \frac{|v|}{1+v^2} = \ln |x| + C$$

$$\frac{|v|}{1+v^2} = e^C |x|$$

$$\frac{\left|\frac{y}{x}\right|}{1+\frac{y^2}{x^2}} = e^C |x|$$

$$\frac{|y|}{x^2+y^2} = e^C$$

$$\boxed{\frac{y}{x^2+y^2} = c\,x}$$

$c \in \mathbb{R}$

($y(x) = 0$ is also solution)

1.6.57

$$\frac{dy}{dx} + P(x)\, y = Q(x)\, y \ln y$$

Let $v = \ln y$

$$\frac{dv}{dx} = \frac{1}{y}\frac{dy}{dx}$$

$$= \frac{1}{y}\left[Q(x)\, y \ln y - P(x)\, y \right]$$

$$= Q(x) \ln y - P(x)$$

$$= Q(x)\, v - P(x)$$

$$\boxed{\frac{dv}{dx} - Q(x)\, v = -P(x)} \qquad (**)$$

1.6.58

$$x \frac{dy}{dx} - 4x^2 y + 2y \ln y = 0$$

$$\frac{dy}{dx} \underbrace{- 4x}_{= P(x)}\, y = \underbrace{-\frac{2}{x}}_{= Q(x)}\, y \ln y$$

Hence, $(**)$ yields

$$\frac{dv}{dx} + \frac{2}{x} v = 4x$$

Linear equation

$$\mu = e^{\int \frac{2}{x} dx}$$

$$= e^{2 \ln x}$$

$$= x^2$$

$$\frac{d}{dx}(\mu v) = 4x \cdot x^2 = 4x^3$$

$$\Rightarrow \quad \mu v = x^4 + c$$

$$\Leftrightarrow \quad v = x^2 + \frac{c}{x^2}$$

Therefore,

$$\ln y = x^2 + \frac{c}{x^2}$$

$$\boxed{y(x) = e^{x^2} e^{\frac{c}{x^2}}}$$

1.6.59

$$\frac{dy}{dx} = \frac{x - y + 1}{x + y + 3}$$

$$\begin{cases} x = u + h \\ y(x) = v(u) + k \end{cases}$$

$$v(u) = y(x) - k$$

$$\frac{dv}{du}(u) = \frac{dy}{dx}(x)\,\frac{dx}{du} = \frac{x - y(x) + 1}{x + y(x) + 3}$$

$$\frac{dv}{du} = \frac{u+h-(v+k)+1}{u+h+(v+k)+3}$$

$$= \frac{u-v+h-k+1}{u+v+h+k+3}$$

We take

$$\begin{cases} h-k+1=0 \\ h+k+3=0 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} k=-1 \\ h=-2 \end{cases}$$

Then,

$$\frac{dv}{du} = \frac{u-v}{u+v}$$

$$\frac{dv}{du} = \frac{1-\frac{v}{u}}{1+\frac{v}{u}}$$

$$\frac{dv}{du} = F(w)$$

for $w = \frac{v}{u}$ and $F(w) = \frac{1-w}{1+w}$

$$uw = v \quad \Rightarrow \quad u\frac{dw}{du} + w = \frac{dv}{du}$$

$$\Leftrightarrow \quad u\frac{dw}{du} = F(w) - w$$

$$\Leftrightarrow \quad u\frac{dw}{du} = \frac{1-w}{1+w} - w$$

$$\Leftrightarrow \quad u \frac{dw}{du} = \frac{1-2w-w^2}{1+w}$$

$$\frac{1+w}{1-2w-w^2}\, dw = \frac{1}{u}\, du$$

$(1-2w-w^2 \neq 0)$

$$\int \frac{1+w}{1-2w-w^2}\, dw = \frac{1}{2}\int \frac{dz}{1-z}$$

$$\begin{cases} z = w^2+2w \\ dz = (2w+2)\, dw \end{cases}$$

$$= -\frac{1}{2} \ln|z-1|$$

$$= -\frac{1}{2}\ln|w^2+2w-1|$$

$$-\frac{1}{2} \ln|w^2+2w-1| = \ln|u| + c$$

$$\ln|w^2+2w-1| = -2\ln|u| + C$$

$$w^2+2w-1 = C\frac{1}{|u|^2} \qquad C \neq 0$$

(we add $C=0$ because $w^2+2w-1=0$ gives also a solution)

$$\left(\frac{v}{u}\right)^2 + 2\left(\frac{v}{u}\right) - 1 = C\frac{1}{u^2}$$

$$\boxed{v^2+2vu-u^2=C}$$

Alternatively,

$$\frac{dv}{du} = \frac{u-v}{u+v}$$

$$(u+v)\,dv - (u-v)\,du = 0$$

$$\frac{\partial(u+v)}{\partial u} = 1$$

$$\frac{\partial(-(u-v))}{\partial v} = 1$$

Hence, the form is exact.

We solve $\begin{cases} \dfrac{\partial F}{\partial u} = -(u-v) \\ \dfrac{\partial F}{\partial v} = u+v \end{cases}$

$$F = -\frac{u^2}{2} + vu + g(v)$$

$$\frac{\partial F}{\partial v} = u + g'(v) = u + v$$

$$\Rightarrow \quad g(v) = \frac{v^2}{2}$$

Hence, $F = -\frac{u^2}{2} + vu + \frac{v^2}{2}$

and the solution satisfies

$$-\frac{u^2}{2} + vu + \frac{v^2}{2} = c$$

$$\boxed{v^2 + 2vu - u^2 = c}$$

Finally

$$(y+1)^2 + 2(y+1)(x+2) - (x+2)^2 = C$$

$$y^2 + 2y + 1 - 2yx - 4y - 2x - 4 - x^2 - 4x - 4 = c$$

$$\boxed{y^2 - 2yx - x^2 - 2y - 6x = c}$$