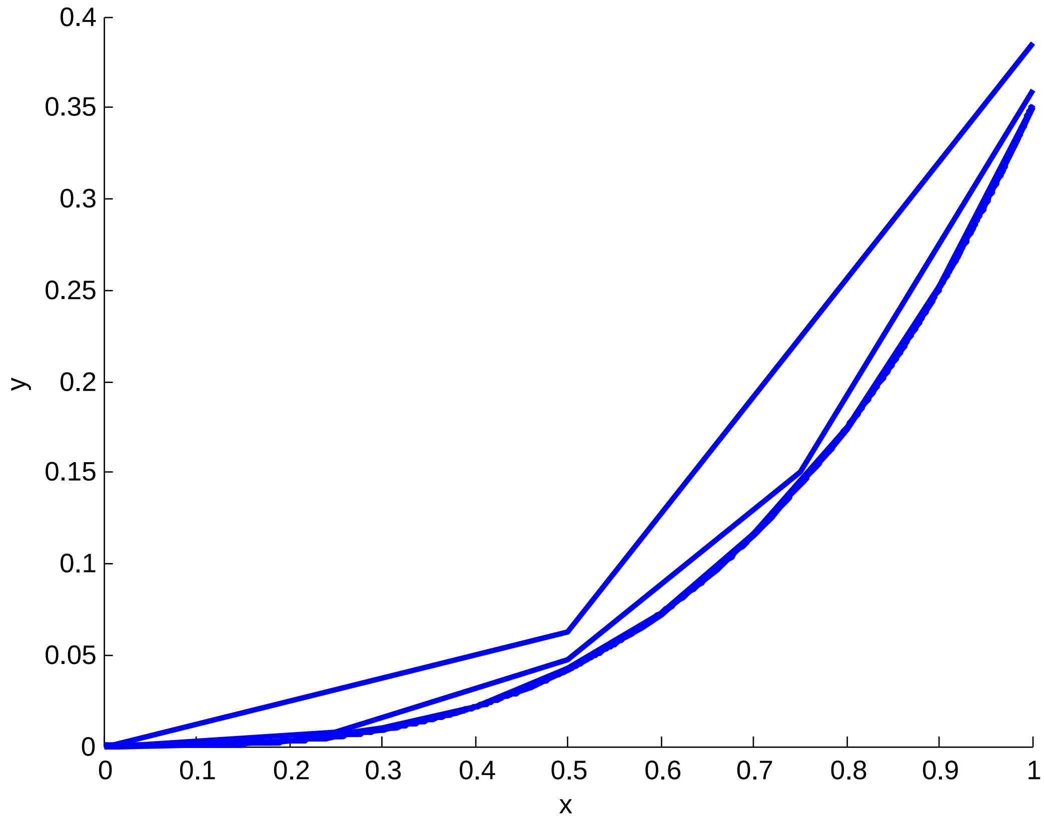
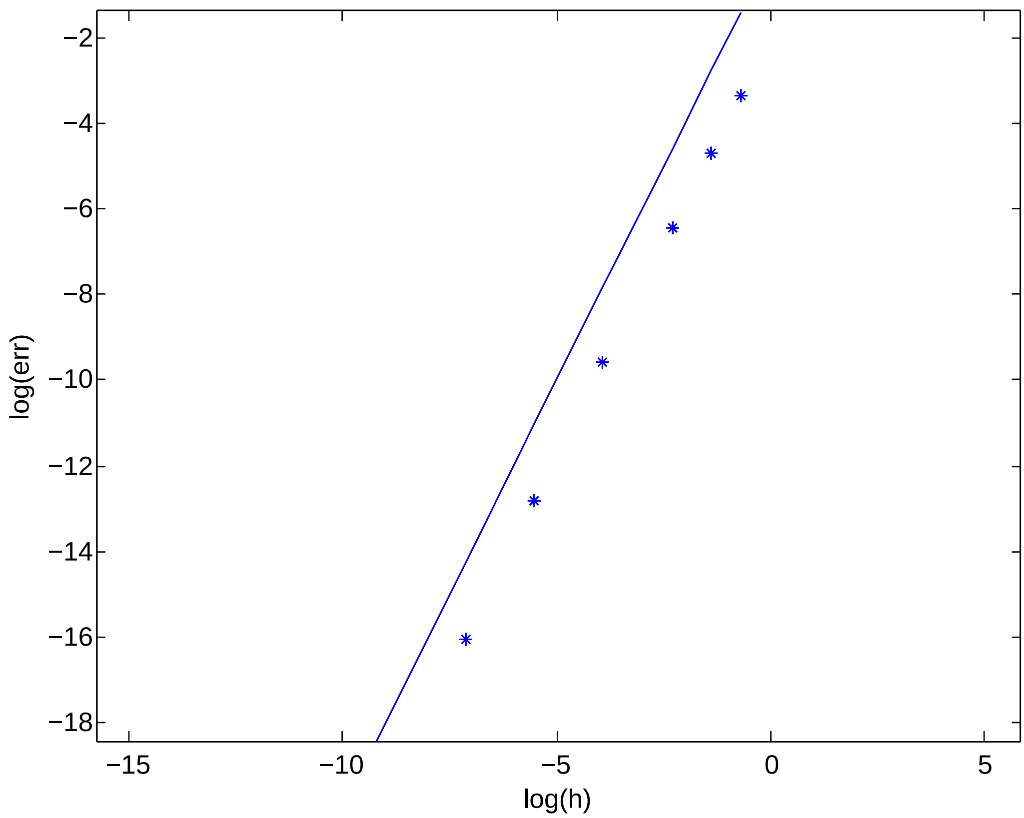
```
function main()

  %Initial value
  y0=0;

  %Array of Step sizes
  hh=[0.5;0.25;0.1;0.02;0.004;0.0008;1e-4];

  %Number of step sizes
  Nh=length(hh);

  %For each step size, we compute the solution
  for i=1:length(hh)

    h=hh(i);

    %Number of points for the given stepsize.
    N=floor(1/h);

    y(1)=y0;

    % Array of x-values where the numerical solution
    % is computed
    x=[0:h:1]';

    for j=1:N
      %We perform the improved euler step
      y(j+1)=improved_euleur_step(x(j),y(j));
    end

    % We store the solution for further use.
    S(i).x=x;
    S(i).y=y;

  end

  % We plot the solutions with different step
  % sizes on the same graph.
  figure(1)
  clf
  set(gcf,'DefaultLinelinewidth',2)
  hold on
  for i=1:Nh
    plot(S(i).x,S(i).y);
  end
  xlabel('x')
  ylabel('y')

  % Create table
  format long
  for i=1:Nh
    y=S(i).y;
    yend(i,1)=y(end);
  end
  yexact=yend(Nh);
  err=abs(yend-yexact);

  figure(2)
  %create table
  t=uitable;
  set(t,'data',[hh yend err])
  set(gcf,'position',[1 1 700 250])

  figure(3)
  % log log plot
  plot(log(hh),log(err),'*')
  hold on
```

```
  plot(log(hh),2*log(hh))
  axis equal
  xlabel('log(h)')
  ylabel('log(err)')

  % Display settings (not important)
  set(t,'columnname',{'step size' 'end value' 'error'})
  set(t,'units','normalized')
  set(t,'position',[0 0 1 1]);
  set(t,'columnformat',{'short' 'long' 'short'});
  set(t,'fontsize',20);

function z=improved_euleur_step(x,y)
  k1=f(x,y);
  y1=y+h*k1;
  k2=f(x+h,y1);
  z=y+h*0.5*(k1+k2);
  end

function z=f(x,y)
  z=x.^2+y.^2;
  end

  end
```


0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
y
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x

	step size	end value	error
1	0.5000	0.384949684143066	0.0347
2	0.2500	0.359432910114621	0.0092
3	0.1000	0.351830132527776	0.0016
4	0.0200	0.350299793806998	6.7948e-05
5	0.0040	0.350234599629644	2.7536e-06
6	8.0000e-04	0.350231954836713	1.0879e-07
7	1.0000e-04	0.350231846044686	0



−2
−4
−6
−8
−10
−12
−14
−16
−18
log(err)
−15
−10
−5
0
5
log(h)

last exercise

1) First we proceed to a Taylor expansion of the exact solution up to fourth order:

$$y(t_1) = y(t_0) + h\, y'(t_0) + \frac{h^2}{2} y''(t_0) + \frac{h^3}{6} y'''(t_0) + \frac{h^4}{24} y''''(\bar{t})$$

for some $\bar{t} \in [t_0, t_1]$.

We have
$$y'(t) = f(y)$$
$$y''(t) = f'(y)\, y' = f'(y) f(y)$$
$$y'''(t) = \left[f''(y) f(y) + f'(y)^2\right] y'$$
$$= f''(y) f(y)^2 + f'(y)^2 f(y)$$

$$y''''(t) = \left[f'''(y) f(y)^2 + 2 f''(y) f(y) f'(y) + 2 f'(y) f''(y) f(y) + f'(y)^3 \right] y'$$
$$= f''' f^3 + 4 f'' f' f^2 + f'^3 f$$

Here the functions are evaluated at y.

so that

(*) $$y(t_1) = y_0 + h f + \frac{h^2}{2} f' f + \frac{h^3}{6}\left[f'' f^2 + f'^2 f\right] + c_1 h^4$$

for a number c_1 which depend on the value of f and its derivative at some point.

In the expression (*), the functions f and its derivatives are evaluated at y_0.

We know look at the computed solution. We have:

$$y_1 = y_0 + \frac{h}{4}\left[f(y_0) + 3f\left(y_0 + \frac{2h}{3} m_{1,2}\right)\right]$$

We use again a Taylor expansion and obtain

(1) $$y_1 = y_0 + \frac{h}{4} f(y_0) + \frac{3h}{4}\left[f(y_0) + \frac{2h}{3} m_{1,2} f'(y_0) + \left(\frac{2h}{3} m_{1,2}\right)^2 \frac{f''(y_0)}{2} + c_2 h^3\right]$$

where the number c_2 depends only on f and the derivatives of f evaluated at some point (We are not interested by the exact value of these numbers as it will be explain later)

(2) $$y_1 = y_0 + hf + \frac{h^2}{2} m_{1,2} f'(y_0) + \frac{h^3}{6} m_{1,2}^2 f''(y_0) + \frac{3c_2}{4} h^4$$

We have $m_{1,2} = f(y_0 + \frac{h}{3} f(y_0))$ and, after taking a Taylor expansion, it gives

(3) $$m_{1,2} = f(y_0) + \frac{h}{3} f(y_0) f'(y_0) + c_3 h^2$$

where the number c_3 again only depends on f and its derivatives. After plugging (3) into (2), we get

$$y_1 = y_0 + hf + \frac{h^2}{2} \cdot \left[f + \frac{h}{3} ff' + c_3 h^2\right] f' + \frac{h^3}{6}\left[f + \frac{h}{3} ff' + c_3 h^2\right]^2 f'' + \frac{3}{4} c_2 h^4$$

4) $$y_1 = y_0 + hf + \frac{h^2}{2} ff' + \frac{h^3}{6}\left[f f'^2 + f'' f^2 \right] + h^4\left[\frac{c_3}{2} f' + \frac{1}{3} f\left[\frac{ff'}{3} + c_3 h\right] + \frac{1}{6}\left[\frac{ff'}{3} + c_3 h\right]^2 + \frac{3}{4} c_2 \right]$$

Now we substract (4) to (*) and the first four terms cancel (y_0 and the terms which involve h, h^2 and h^3 disappear). We are left with the terms in h^4

$$y_1 - y(t_1) = c_4 h^4$$

where c_4 is a "big term" which involves f and its derivatives and h. Since the derivatives and f are bounded by M and h is smaller than one, we conclude that $|c_4| \leq C(M)$ for some function of M and therefore

$$|y_1 - y(t_1)| \leq C(M) h^4$$

2) The order of the method is $\underline{\underline{3}}$ because the local error has order 4 and we lose 1 degree in the order by cumulating the local errors.