Ex 5.1.10

$$\begin{cases} x'' = (1-y)x \\ y'' = (1-x)y \end{cases}$$

Let
$$\begin{cases} x_1 = x \\ x_2 = x' \\ x_3 = y \\ x_4 = y' \end{cases}$$

$$\begin{cases} x_1' = x_2 \\ x_2' = (1-x_3)x_1 \\ x_3' = x_4 \\ x_4' = (1-x_1)x_2 \end{cases}$$

Ex. 5.10.21

a) $$\frac{d}{dt}\left[x^2(t) + y^2(t)\right] = 2x\dot{x} + 2y\dot{y}$$
$$= 2xy + 2y(-x)$$
$$= 0$$

Hence
$$x^2(t) + y^2(t) = C \quad , \quad C \text{ is a constant}$$

and the trajectories are circles with center at origo.

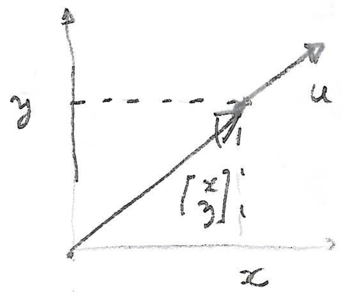
b) $$\frac{d}{dt}\left[x^2 - y^2\right] = 2x\dot{x} - 2y\dot{y} = 2xy - 2xy = 0$$

Hence,

$$x^2(t) - y^2(t) = C$$

and the trajectories are hyperbolas with axis $y = \pm x$.

5.1.29



$$u = \frac{\begin{bmatrix} x \\ y \end{bmatrix}}{\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|}$$ unit vector

The force is given by

$$\vec{F} = -\frac{k}{(x^2+y^2)}\,\vec{u}$$

$$\vec{F} = -\frac{k}{(x^2+y^2)^{3/2}} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$m\vec{a} = \vec{F} \quad \text{(Newton's law)}$$

gives

$$\begin{cases} mx'' = -\dfrac{kx}{(x^2+y^2)^{3/2}} \\[2ex] my'' = -\dfrac{ky}{(x^2+y^2)^{3/2}} \end{cases}$$

Extra exercise

a) equilibrium:

$$\begin{cases} x^2 - 2xy = 0 \\ 2xy - y^2 = 0 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} x = 0 \quad \text{or} \quad y = \frac{1}{2}x \\ y = 0 \quad \text{or} \quad y = 2x \end{cases}$$

$$\Leftrightarrow \quad x = y = 0$$

vertical isoclines:

$$\frac{dx}{dt} = 0 \quad \Leftrightarrow \quad x^2 - 2xy = 0$$

$$\Leftrightarrow \quad x = 0 \quad \text{or} \quad y = \frac{1}{2}x .$$

horizontal isoclines:

$$\frac{dy}{dt} = 0 \quad \Leftrightarrow \quad 2xy - y^2 = 0$$

$$\Leftrightarrow \quad y = 0 \quad \text{or} \quad y = 2x$$

b)

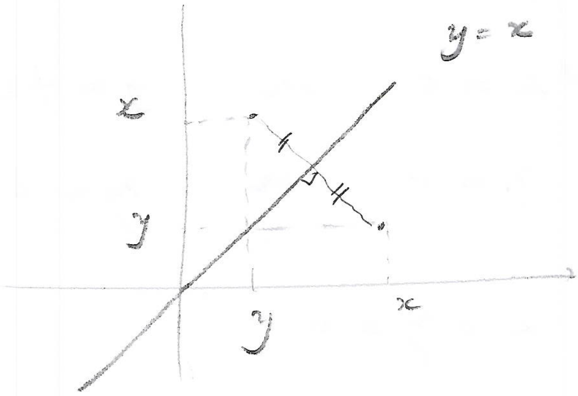
$$F\begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} x^2 - 2x^2 \\ 2x^2 - x^2 \end{bmatrix} = \begin{bmatrix} -x^2 \\ x^2 \end{bmatrix} = x^2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

the vector field allways points in the direction of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ on $y = x$.

$$F\begin{bmatrix} x \\ -x \end{bmatrix} = \begin{bmatrix} x^2 + 2x^2 \\ -2x^2 - x^2 \end{bmatrix} = 3x^2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The vector fields allways point in the direction of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ on $y=-x$.

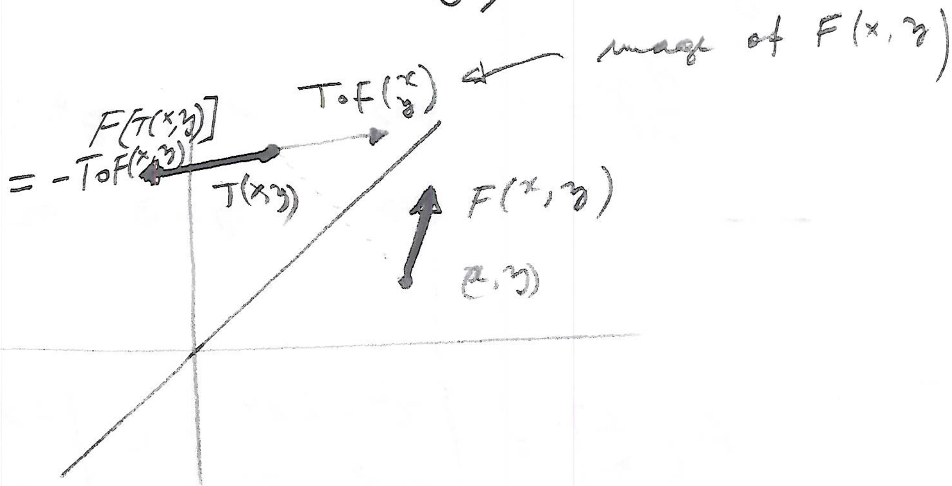
c)



reflexion with respect to line $y=x$.

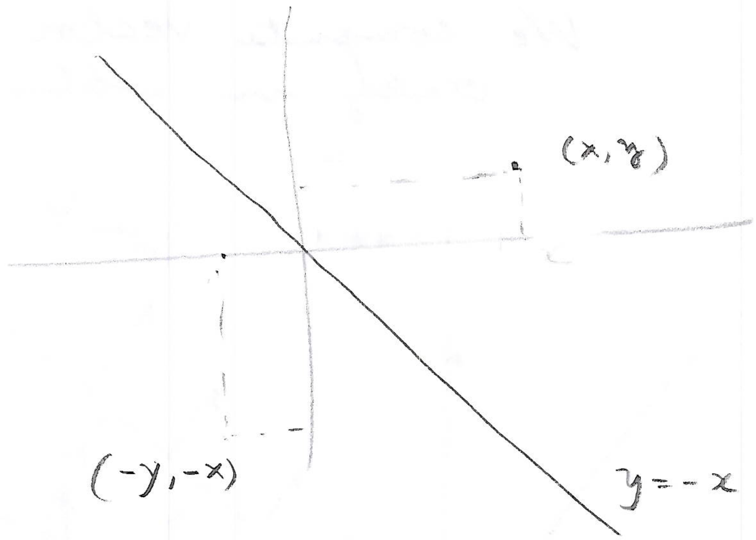
$$F\circ T\begin{pmatrix} x \\ y \end{pmatrix} = F\begin{pmatrix} y \\ x \end{pmatrix} = \begin{matrix} y^2 - 2xy \\ 2xy - x^2 \end{matrix} = -\begin{bmatrix} 2xy - y^2 \\ x^2 - 2xy \end{bmatrix}$$

$$= -\,T\circ F\begin{pmatrix} x \\ y \end{pmatrix}$$



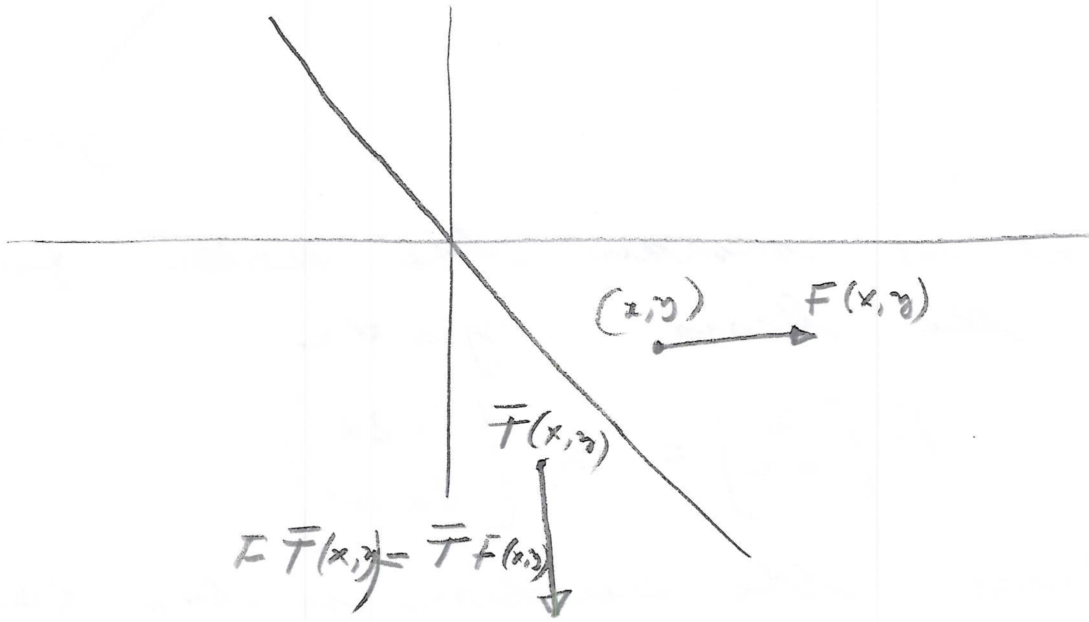
Conclusion: it is enough to know how the vector field looks like on one side of the line $y=x$.

d)



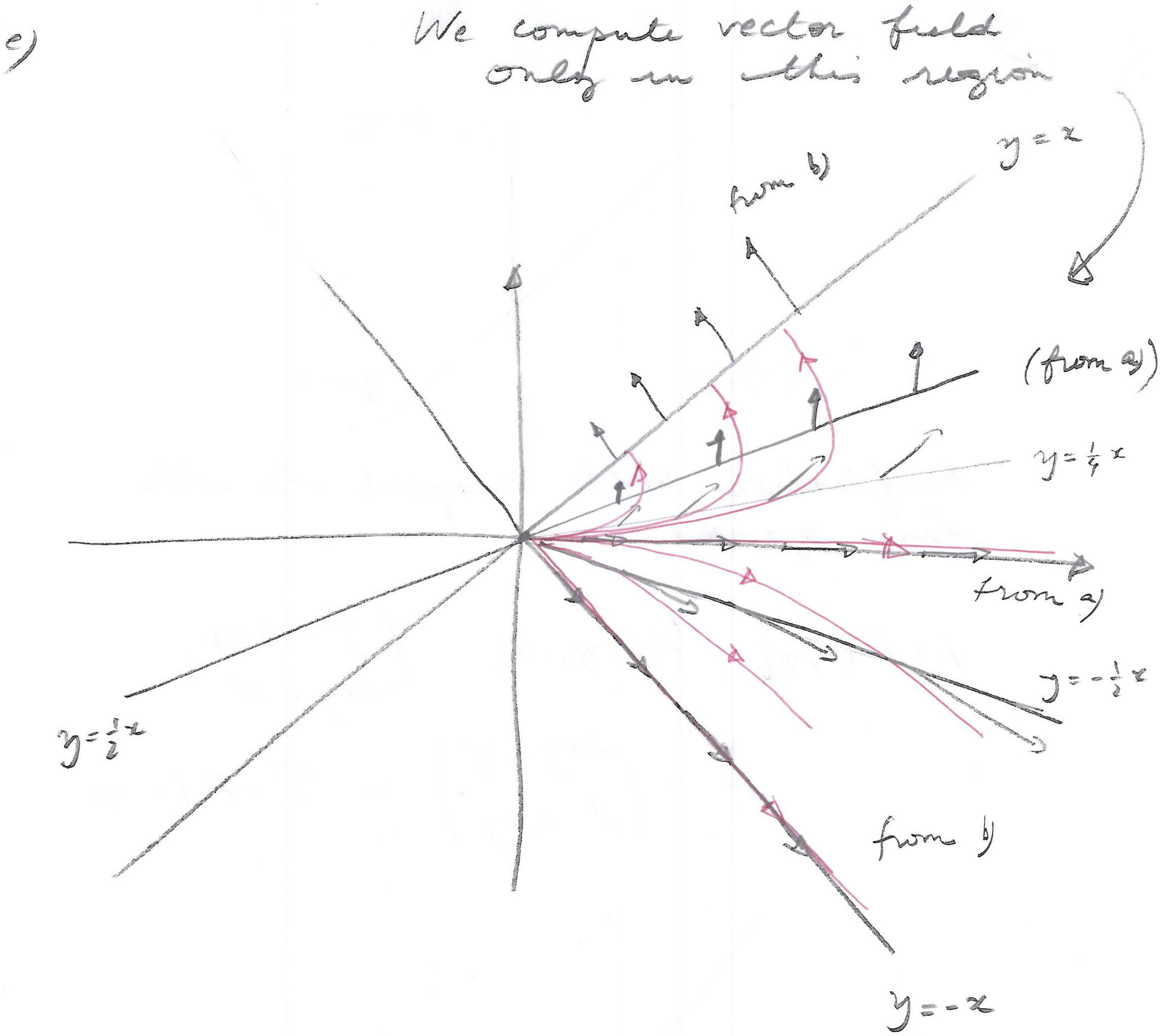
Reflection with respect to the line $y=-x$.

$$F\circ\bar{F}(x,y) = F(-y,-x) = \begin{pmatrix} y^2-2xy \\ 2xy-x^2 \end{pmatrix}$$
$$= -\begin{pmatrix} 2xy-y^2 \\ x^2-2xy \end{pmatrix} = \bar{F}\circ F(x,y)$$



Conclusion: we only need to compute the vector field on one side of the line $y=-x$.

c)



Let us consider the vector field on the lines $y = \alpha x$

$$F\begin{bmatrix} x \\ \alpha x \end{bmatrix} = x^2 \begin{bmatrix} 1 - 2\alpha \\ 2\alpha - \alpha^2 \end{bmatrix}$$

Hence, the direction is the same (amplitude is increasing in x)