Exercise 5.4.6

$$\begin{cases} x_1' = 9x_1 + 5x_2 \\ x_2' = -6x_1 - 2x_2 \end{cases}$$

$x_1(0) = 1$, $x_2(0) = 0$

$X' = AX$ $\qquad A = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix}$

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

We compute the eigenvalues and eigenvectors of A.

$$|A - \lambda I| = \begin{vmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{vmatrix} = (9-\lambda)(-2-\lambda) + 30$$

$$= \lambda^2 + 2\lambda - 9\lambda + 30 - 18$$

$$= \lambda^2 - 7\lambda + 12$$

[In general, we have

$$|A - \lambda I| = \lambda^2 - \underbrace{\operatorname{tr} A}_{=7} \lambda + \underbrace{\det A}_{=12}$$]

$$\lambda^2 - 7\lambda + 12 = 0$$

Let $\Delta = 49 - 4 \cdot 12 = 1$

Hence, $\lambda = \dfrac{7 \pm 1}{2} = \begin{cases} 4 \\ 3 \end{cases}$

We compute the eigenvector space of $\lambda = 4$ (1)

$$\begin{bmatrix} 5 & 5 \\ -6 & -6 \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = 0 \iff \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

Let $u_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $\lambda = 3$,

$$\begin{bmatrix} 6 & 5 \\ -6 & -5 \end{bmatrix}\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = 0 \iff \begin{bmatrix} 6 & 5 \end{bmatrix}\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = 0$$

Let $u_2 = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$

The eigenvalues are distinct

$\Rightarrow$ They form a basis

$\Rightarrow$ The general solution is

$$X(t) = c_1 e^{\lambda_1 t} u_1 + c_2 e^{\lambda_2 t} u_2$$
$$= c_1 e^{4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 5 \\ -6 \end{bmatrix}$$

for any constant c_1, c_2

$$X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \iff c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 5 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\iff \begin{bmatrix} 1 & 5 \\ -1 & -6 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We have to solve this linear system (Use the method of your choice)

$$\left[\begin{array}{cc|c} 1 & 5 & 1 \\ -1 & -6 & 0 \end{array}\right] \longleftrightarrow \left[\begin{array}{cc|c} 1 & 5 & 1 \\ 0 & -1 & 1 \end{array}\right] \begin{array}{l} (1)' = (1) \\ (1)' = (1) + 2 \end{array}$$

$$\longleftrightarrow \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & -1 \end{array}\right] \begin{array}{l} (1)' = (1) + (2).5 \\ (2)' = -(2) \end{array}$$

The solution is

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

and

$$X(t) = 6e^{4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{3t}\begin{bmatrix} 5 \\ -6 \end{bmatrix}$$

$$\begin{cases} x_1(t) = 6e^{4t} - 5e^{3t} \\ x_2(t) = -6e^{4t} + 6e^{3t} \end{cases}$$

<u>5.4.19</u>

$$\begin{cases} x_1' = 4x_1 + x_2 + x_3 \\ x_2' = x_1 + 4x_2 + x_3 \\ x_3' = x_1 + x_2 + 4x_2 \end{cases}$$

$$X' = AX \quad \text{where} \quad A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

$$\text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Note that the sum of the columns are constant

$$\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence, $\lambda_1 = 6$ is eigenvalue for which $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is eigenvector.

$$\begin{vmatrix} 4-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

for $\lambda = 3$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \\ u_{23} \end{bmatrix} = 0$$

$$\Leftrightarrow \quad \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \\ u_{23} \end{bmatrix} = 0$$

The dimension of the eigenvector space is <u>**2**</u>.

Let $u_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\bar{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

We have

$$X(t) = c_1 e^{\lambda_1 t} u_1 + c_2 e^{\lambda_2 t} u_2 + c_3 e^{\lambda_2 t} \overline{u_2}$$
$$= c_1 e^{6t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + e^{3t}\left(c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right)$$

for any constant $c_1, c_2, c_3 \in \mathbb{R}^3$

Ex 5.4.41

$$\begin{cases} x_1' = 4x_1 + x_2 + x_3 + 7x_4 \\ x_2' = x_1 + 4x_2 + 10x_3 + x_4 \\ x_3' = x_1 + 10x_2 + 4x_3 + x_4 \\ x_4' = 7x_1 + x_2 + x_3 + 4x_4 \end{cases}$$

$$A = \begin{bmatrix} 4 & 1 & 1 & 7 \\ 1 & 4 & 10 & 1 \\ 1 & 10 & 4 & 1 \\ 7 & 1 & 1 & 4 \end{bmatrix}$$

$$|A + 3I| \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \end{bmatrix} = 0 \iff \underbrace{\begin{bmatrix} 7 & 1 & 1 & 7 \\ 1 & 7 & 10 & 1 \\ 1 & 10 & 7 & 1 \\ 7 & 1 & 1 & 7 \end{bmatrix}}_{\text{matrix of rank 3}} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ kernel is of rank 1

Let $B = \begin{bmatrix} 7 & 1 & 1 \\ 1 & 7 & 10 \\ 1 & 10 & 7 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} & B & & \begin{matrix}7\\1\\4\end{matrix} \\ \hline 7 & 1 & 1 & 7 \end{array}\right]\begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix} = 0$$

$$\iff \underbrace{\left[\begin{array}{ccc|c} & B^{-1} & & \begin{matrix}0\\0\\0\end{matrix} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]}_{\text{invertible matrix}}\left[\begin{array}{ccc|c} & B & & \begin{matrix}7\\1\\4\end{matrix} \\ \hline 7 & 1 & 1 & 7 \end{array}\right]\begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Matrix block multiplication

$$\iff \left[\begin{array}{ccc|c} & Id & & v \\ \hline 7 & 1 & 1 & 7 \end{array}\right]\begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

where $v = B^{-1}\begin{bmatrix}7\\1\\4\end{bmatrix}$

$$\iff \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & \end{array}\right]\begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\iff \begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} + u_4\, v = 0$$

Let $\alpha = -u_4$ we have

$$\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} = \alpha\, v$$

$$u_4 = -\alpha$$

An eigenvector is given by $u = \left[\begin{array}{c} v \\ \hline -1 \end{array}\right]$

Conclusion: we have to solve

$$Bv = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 7 & 1 & 1 & 7 \\ 1 & 7 & 10 & 1 \\ 1 & 10 & 7 & 1 \end{array}\right] \longleftrightarrow \left[\begin{array}{ccc|c} 1 & \frac{1}{7} & \frac{1}{7} & 1 \\ 0 & \frac{48}{7} & \frac{69}{7} & 0 \\ 0 & \frac{69}{7} & \frac{48}{7} & 0 \end{array}\right] \begin{array}{l} (1') = \frac{1}{7}(1) \\ (2') = (2) - \frac{1}{7}(1) \\ (3') = (3) - \frac{1}{7}(1) \end{array}$$

$$\longleftrightarrow \left[\begin{array}{ccc|c} 1 & \frac{1}{7} & \frac{1}{7} & 1 \\ 0 & 1 & \frac{69}{48} & 0 \\ 0 & 0 & \frac{117}{16} & 0 \end{array}\right] \begin{array}{l} (2') = \frac{7}{48}(2) \\ (3') = (3) - \frac{69}{7}\cdot\frac{7}{48}(2) \end{array}$$

$\Rightarrow \quad v_3 = 0$

$v_2 + \frac{117}{16} v_3 = 0 \quad \Longrightarrow \quad v_2 = 0$

and $\quad v_1 + \frac{1}{7} v_2 + \frac{1}{7} v_3 = 1$

$\Rightarrow \quad v_1 = 1$

Hence $\quad u = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ is an eigenvector.

For $\lambda = -6$

$$A + 6I = \begin{bmatrix} 10 & 1 & 1 & 7 \\ 1 & 10 & 10 & 1 \\ 1 & 10 & 10 & 1 \\ 7 & 1 & 1 & 10 \end{bmatrix}$$

Let $B = \begin{bmatrix} 10 & 1 & 1 \\ 1 & 10 & 10 \\ 1 & 10 & 10 \end{bmatrix}$

but $\det B = 0$, B is not invertible.

$$[A + 6I]\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = 0 \iff \begin{bmatrix} 10 & 1 & 7 & 1 \\ 1 & 10 & 1 & 10 \\ 7 & 1 & 10 & 1 \\ 1 & 10 & 1 & 10 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_4 \\ u_3 \end{bmatrix} = 0$$

we exchange columns 3 and 4 and rows 3 and 4

let

$$B = \begin{bmatrix} 10 & 1 & 7 \\ 1 & 10 & 1 \\ 7 & 1 & 1 \end{bmatrix}$$

$$\det B = 504 \neq 0$$

we want to solve $Bv = \begin{bmatrix} 1 \\ 10 \\ 1 \end{bmatrix}$

$$v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

we take as eigenvector

$$u = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

For $\lambda = 10$,

$$A - 10I = \begin{bmatrix} -6 & 1 & 1 & 7 \\ 1 & -6 & 10 & 1 \\ 1 & 10 & -6 & 1 \\ 7 & 1 & 1 & -6 \end{bmatrix}$$

$$B = \begin{bmatrix} -6 & 1 & 1 \\ 1 & -6 & 10 \\ 1 & 10 & -6 \end{bmatrix}$$

$\det B = 416 \neq 0$

we want to solve

$$Bv = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}$$

we get $\quad v = \begin{bmatrix} -1 \\ 1/2 \\ 1/2 \end{bmatrix}$

We take $\quad u = \begin{bmatrix} -2 \\ 1 \\ 1 \\ -2 \end{bmatrix}$ as eigenvector.

For $\lambda = 15$

$$A - 15I = \begin{bmatrix} -11 & 1 & 1 & 7 \\ 1 & -11 & 10 & 1 \\ 1 & 10 & -11 & 1 \\ 7 & 1 & 1 & -11 \end{bmatrix}$$

Let $\quad B = \begin{bmatrix} -11 & 1 & 1 \\ 1 & -11 & 10 \\ 1 & 10 & -11 \end{bmatrix}$

$\det B = -189 \neq 0$

We solve $\quad Bv = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}$

$$v = \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix}$$

We take

$$u = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \text{ as eigenvector}$$

Conclusion: The general solution of the ODE is

$$X(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + c_3 e^{10t} \begin{bmatrix} -2 \\ 1 \\ 1 \\ 2 \end{bmatrix} + c_4 e^{15t} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

$$X(0) = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 1 & 2 \\ -1 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

$$\Longrightarrow \quad \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ -5 \\ 3 \end{bmatrix}.$$