Ex 5.4.11

$$\begin{cases} x_1' = x_1 - 2x_2 \\ x_2' = 2x_1 + x_2 \end{cases}$$

$$\dot{X} = AX \quad \text{where} \quad A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \lambda^2 - 2\lambda + 5 = 0$$

$$\Delta = 4 - 5 \cdot 4 = -16$$

We have a complex eigenvalue

$$\lambda = \frac{2 + i4}{2} = 1 + 2i$$

We compute the eigenvector

$$\begin{bmatrix} -2i & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\Longrightarrow \quad u = \begin{bmatrix} 1 \\ -i \end{bmatrix} \text{ is eigenvector.}$$

The complex solution is given by

$$X(t) = c_1 e^{\lambda t} u + c_2 e^{\bar{\lambda} t} \bar{u}$$

for any $c_1, c_2 \in \mathbb{C}$

The real solution is given by

$$X(t) = c_1 \operatorname{Re}[e^{\lambda t} u] + c_2 \operatorname{Im}[e^{\lambda t} u]$$

for any $c_1, c_2 \in \mathbb{R}$.

$$e^{\lambda t} u = e^{t}(\cos(2t) + i\sin(2t)) \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= e^{t}\left(\begin{bmatrix} \cos(2t) \\ \sin 2t \end{bmatrix} + i \begin{bmatrix} \sin(2t) \\ -\cos 2t \end{bmatrix} \right)$$

Hence,

$$X(t) = c_1 e^{t} \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix} + c_2 e^{t} \begin{bmatrix} \sin 2t \\ -\cos 2t \end{bmatrix}$$

$$X(0) = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

that is, $c_1 = 0$, $c_2 = -4$

$$X(t) = -4e^{t} \begin{bmatrix} \sin(2t) \\ -\cos(2t) \end{bmatrix}.$$

ex 5.4.15

$$\begin{cases} x_1' = 7x_1 - 5x_2 \\ x_2' = 4x_1 + 3x_2 \end{cases}$$

$\dot{X} = AX$ where $A = \begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix}$

$$|A - \lambda I| = \lambda^2 - 10\lambda + 41$$

We have a complex conjugate pair of eigenvalues

$$\lambda = 5 + 4i$$

We compute an eigenvector

$$\begin{bmatrix} 2-4i & -5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\Rightarrow \quad u = \begin{bmatrix} 5 \\ 2-4i \end{bmatrix}$$ is an eigenvector

$$e^{\lambda t} u = e^{5t} \left[\cos(4t) + i \sin(4t) \right] \begin{bmatrix} 5 \\ 2-4i \end{bmatrix}$$

$$= e^{5t} \left(\begin{bmatrix} 5\cos(4t) \\ 2\cos(4t) + 4\sin(4t) \end{bmatrix} + i \begin{bmatrix} 5\sin(4t) \\ -4\cos(4t) + 2\sin(4t) \end{bmatrix} \right)$$

The general solution is

$$X(t) = c_1 e^{5t} \begin{bmatrix} 5\cos(4t) \\ 2\cos(4t) + 4\sin(4t) \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 5\sin(4t) \\ -4\cos(4t) + 2\sin(4t) \end{bmatrix}$$

Ex 5.4.29

$$\begin{cases} \dfrac{dx_1}{dt} = -k_1 x_1 + k_2 x_2 \\ \dfrac{dx_2}{dt} = k_1 x_1 - k_2 x_2 \end{cases}$$

$$\dot{X} = AX \quad \text{where} \quad A = \begin{bmatrix} -k_1 & k_2 \\ k_1 & -k_2 \end{bmatrix}$$

Note that $\frac{d}{dt}(x_1 + x_2) = 0$

$\Longrightarrow$ The total amount of salt is conserved.

We have:
$$\begin{cases} r = 10 & (\text{gal/min}) \\ V_1 = 50 & (\text{gal}) \\ V_2 = 25 & (\text{gal}) \end{cases}$$

Hence,
$$k_1 = \frac{r}{V_1} = \frac{10}{50} = \frac{1}{5},$$

$$k_2 = \frac{r}{V_2} = \frac{10}{25} = \frac{2}{5}.$$

$$A = \frac{1}{5} \underbrace{\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}}_{= A_0}$$

$$|A - \lambda I| = \left|\frac{1}{5} A_0 - \lambda I\right| = \left(\frac{1}{5}\right)^2 |A_0 - 5\lambda I|$$

$$|A_0 - \lambda_0 I| = 0 \iff \lambda_0^2 + 3\lambda_0 = 0$$

$$\iff \lambda_0(\lambda_0 + 3) = 0$$

Two distinct eigenvalues

$$\lambda_{01} = 0 \qquad \lambda_{02} = -3$$

We compute the eigenvectors.

For $\lambda_{01} = 0$

$$[-1, \ 2]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\Longrightarrow \quad u = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{is an eigenvector}$$

for $\lambda_{0,2} = -3$,

$$[2 \quad 2]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\Longrightarrow \quad u = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{is eigenvector}$$

We have

$$A_0 u = \lambda_0 u \iff \frac{1}{5} A_0 u = \frac{\lambda_0}{5} u$$

$$\iff A u = \lambda u$$

$$(5\lambda = \lambda_0)$$

Hence, eigenvectors for A_0 and A are the same.

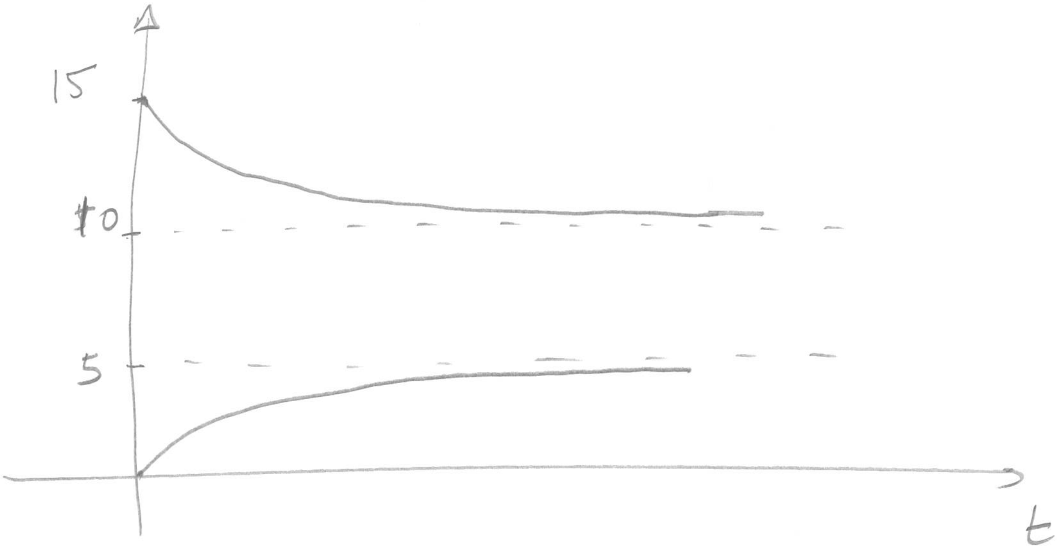
The general solution is

$$X(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$X(0) = \begin{bmatrix} 15 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \end{bmatrix}$$

$$\iff \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$X(t) = \begin{bmatrix} 10 + 5e^{-3t} \\ 5 - 5e^{-3t} \end{bmatrix}$$



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