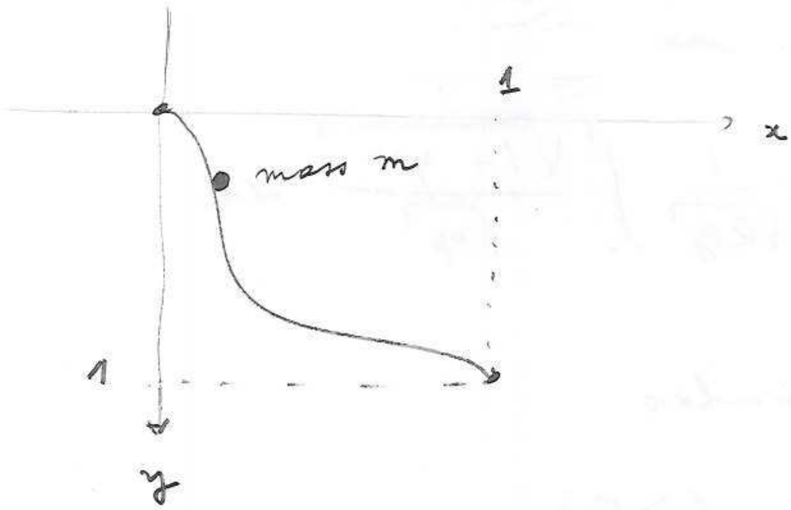
The brachistone problem



Let us prove (1) (not asked in the exercise)

The sum of the kinetic energy $\frac{1}{2}mv^2$ and potential energy $-mgy$ is preserved

Hence,
$$\frac{1}{2}mv^2 - mgy = \frac{1}{2}mv_0^2 - mgy_0 \quad \text{(at time } t=0\text{)}$$
$$= 0$$

and
$$v = \sqrt{2gy}$$

We have $\quad \frac{ds}{dt} = v \quad$ (definition of velocity v)

Since $\quad ds = \sqrt{1+y'(x)^2}\,dx$, we get

$$dt = \frac{\sqrt{1+y'^2(x)}}{\sqrt{2g\,y(x)}}\,dx$$

After integration, we obtain that the time used by the particle to reach $(1,1)$ is

$$T(y) = \frac{1}{\sqrt{2g}} \int_0^1 \frac{\sqrt{1+y'^2}}{\sqrt{y}}\, dx$$

The EL writes

$$\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) = 0$$

$$\Leftrightarrow \quad -\frac{1}{2} y^{-3/2}(1+y'^2)^{1/2} - \frac{d}{dt}\left(y^{-1/2}(1+y'^2)^{-1/2} y'\right) =$$

complicated eq^n! We notice that F does not depend on time. In this case, there allways exists a first integral which is given by

$$F - \frac{\partial F}{\partial y'} y' = C \qquad \text{for some constant } C.$$

$$\frac{1}{\sqrt{2g}}\left(\frac{\sqrt{1+y'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y}\sqrt{1+y'^2}}\right) = c$$

$$\frac{1}{\sqrt{y(1+y'^2)}} = c\sqrt{2g}$$

$$\Leftrightarrow \qquad 1 + y'^2 = \frac{c}{y} \qquad (\text{for another constant } c)$$

$$\Leftrightarrow \qquad y' = \sqrt{\frac{c-y}{y}}$$

Hence

$$x = \int_0^y \sqrt{\frac{\bar{y}}{c - \bar{y}}}\, d\bar{y}$$

Let

$$\bar{v} = \sqrt{\frac{\bar{y}}{c}}$$

Then, $d\bar{y} = 2c\,\bar{v}\,d\bar{v}$

and

$$x = \int_0^v \frac{2c\,\bar{v}^2 \sqrt{c}}{\sqrt{c(1-\bar{v}^2)}}\, d\bar{v}$$

$$= 2c \int_0^v \frac{\bar{v}^2}{\sqrt{1-\bar{v}^2}}\, d\bar{v}$$

Let $\bar{v} = \sin\bar{\theta}$

$$d\bar{v} = \cos\bar{\theta}\, d\bar{\theta}$$

$$x = 2c \int_0^\theta \frac{\sin^2\bar{\theta}}{\sqrt{1-\sin^2\bar{\theta}}} \cos\bar{\theta}\, d\bar{\theta}$$

$$= 2c \int_0^\theta \sin^2\bar{\theta}\, d\bar{\theta} = 2c \int_0^\theta \frac{\cos 2\bar{\theta} - 1}{2}\, d\bar{\theta}$$

$$= \frac{c}{2}\left[\sin 2\theta - 2\theta\right]$$

At the same time, we have

$$y(\theta) = c\,v^2 = c\sin^2\theta = \frac{c}{2}\left[1 - \cos 2\theta\right]$$

By reparametrizing the curve by $2\theta \to -\theta$, we obtain that $(x(\theta), y(\theta))$ is a cycloide

We have to determine a

$$\begin{cases} x(\theta) = a(\theta - \sin\theta) \\ y(\theta) = a(1-\cos\theta) \end{cases}$$

for $\theta = 0$, we have $x(0) = y(0) = 0$

$$\begin{cases} x(\theta) = 1 \\ y(\theta) = 1 \end{cases} \iff \begin{cases} a(\theta - \sin\theta) = 1 \\ a(1-\cos\theta) = 1 \end{cases}$$

$$\theta - \sin\theta = 1 - \cos\theta$$

$$\theta = 1 + \sin\theta - \cos\theta$$

Let $g(\theta) = \theta - 1 - \sin\theta + \cos\theta$

$$= \theta - 1 + \frac{2}{\sqrt{2}}\left[\cos\frac{\pi}{4}\cos\theta - \sin\frac{\pi}{4}\sin\theta\right]$$

$$= \theta - 1 + \frac{2}{\sqrt{2}}\cos\left(\theta + \frac{\pi}{4}\right)$$

$$g'(\theta) = 1 - \frac{2}{\sqrt{2}}\sin\left(\theta + \frac{\pi}{4}\right)$$

$$g'(\theta) = 0 \iff \sin\left(\theta + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$\iff \sin\left(\theta + \frac{\pi}{4}\right) = \sin\frac{\pi}{4}$$

$$\iff \begin{cases} \theta + \frac{\pi}{4} = \frac{\pi}{4} + 2k\pi \\ \theta + \frac{\pi}{4} = \frac{3\pi}{4} + 2k\pi \end{cases} \quad k \in \mathbb{N}$$

$$\iff \begin{cases} \theta = 2k\pi \\ \theta = \frac{\pi}{2} + 2k\pi \end{cases}$$

θ	0		$\frac{\pi}{2}$		θ_0		2π
$g'(\theta)$	0	$-$	0	$-$		$+$	0
$g(\theta)$	0	↘	$\frac{\pi}{2}-2$	↗	0	↗	2π

There exists a unique $\theta_0 \in [\frac{\pi}{2}, 2\pi]$ such that $g(\theta_0) = 0$

For $\theta \geqslant 2\pi$, we have

$$g(\theta) \geqslant 2\pi - 3 \geqslant 0$$

Hence θ_0 is the unique positive solution of $\theta - \sin\theta = 1 - \cos\theta$.

We set $$a = \frac{1}{1-\cos\theta_0}$$

Thus,

$$\begin{cases} x(\theta) = \dfrac{\theta - \sin\theta}{\theta_0 - \sin\theta_0} \\[2ex] y(\theta) = \dfrac{1 - \cos\theta}{1 - \cos\theta_0} \end{cases}$$

is the unique solution of the EL equation