Obligatory exercise for MAT2440
Spring 2015

Deadline: Wednesday 15 April.

All the problems can be solved without the aid of a computer or calculator. However you may also use any computer system, like Matlab, Mathematica, Maple, or the numpy package of Python. Whenever doing so, you should document the commands you are using and explain in detail how you apply the results.

Problem 1

Let

$$P(x, y) = 5x^4y - y^5, \qquad \text{and} \qquad Q(x, y) = x^5 - 5xy^4.$$

Solve the differential equation

$$Pdx + Qdy = 0$$

as an exact equation.

Answer: Is the equation exact? Yes, because

$$\frac{\partial P}{\partial y} = 5x^4 - 5y^4 = \frac{\partial Q}{\partial x}.$$

Then there is a solution of the form $F(x, y) = C$, where

$$F = \int P\,dx + g(y), \qquad \frac{\partial F}{\partial y} = Q.$$

So

$$F = x^5y - xy^5 + g(y),$$

and

$$\frac{\partial F}{\partial y} = x^5 - 5xy^4 + g'(y) = Q = x^5 - 5xy^4,$$

and so $g'(y) = 0$ and $g(y) = C_2$ for some constant C_2. Therefore, the solution has the form

$$x^5 y - xy^5 + C_2 = C,$$

in other words,

$$x^5 y - xy^5 = K,$$

for some constant K.

Problem 2

Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}.$$

Find a general solution to the differential equation $\mathbf{x}' = A\mathbf{x}$ using either (a) the method of elimination (write the equation as a system of linear first-order equations), or (b) the eigenvalue method.

Answer (a): Write the system as

$$\begin{aligned} x_1' &= x_1 - x_2, \\ x_2' &= x_1 + 2x_2. \end{aligned}$$

The first equation implies $x_2 = x_1 - x_1'$, and differentiating gives $x_2' = x_1' - x_1''$. Substituing these into the second equation gives

$$(x_1' - x_1'') = x_1 + 2(x_1 - x_1'),$$

or

$$x_1'' - 3x_1' + 3x_1 = 0.$$

The charc. equation is

$$r^2 - 3r + 3 = 0,$$

with roots $r = (2 \pm i\sqrt{3})/2$. So

$$x_1 = e^{(3/2)t}(c_1 c + c_2 s), \qquad (1)$$

where

$$c = \cos\frac{\sqrt{3}}{2}t, \qquad s = \sin\frac{\sqrt{3}}{2}t. \tag{2}$$

To find x_2,

$$x_1' = (3/2)e^{(3/2)t}(c_1 c + c_2 s) + e^{(3/2)t}(-\frac{\sqrt{3}}{2}c_1 s + c_2\frac{\sqrt{3}}{2}c),$$

and so

$$x_2 = x_1 - x_1' = e^{(3/2)t}((-c_1 - \sqrt{3}c_2)c + (\sqrt{3}c_1 - c_2)s)/2. \tag{3}$$

Answer (b): Eigenvalues:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 3.$$

So $\lambda = (3 \pm i\sqrt{3})/2$. It is sufficient to use $\lambda = (3 + i\sqrt{3})/2$. Its eigenvector $\mathbf{v}$ satisfies

$$(A - \lambda I)\mathbf{v} = 0,$$

i.e.,

$$\begin{bmatrix} -1/2 - i\sqrt{3}/2 & -1 \\ 1 & 1/2 - i\sqrt{3}/2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0,$$

and so we can let $\mathbf{v} = [2, -1 - i\sqrt{3}]^T$. Then

$$\mathbf{x} = \mathbf{v}e^{\lambda t} = \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ -\sqrt{3} \end{bmatrix} i \right) e^{(3/2)t}(c + is),$$

with c and s as in (2). So,

$$\mathbf{x}(t) = \mathbf{x}_1(t) + i\mathbf{x}_2(t),$$

where

$$\mathbf{x}_1(t) = e^{(3/2)t}\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix} c - \begin{bmatrix} 0 \\ -\sqrt{3} \end{bmatrix} s \right),$$

$$\mathbf{x}_2(t) = e^{(3/2)t}\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix} s + \begin{bmatrix} 0 \\ -\sqrt{3} \end{bmatrix} c \right).$$

Then the general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

with x_1 and x_2 the same as (1) and (3) if the constants c_1 and c_2 are divided by 2.

Problem 3

Find a general solution to the differential equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 3 & -3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Answer: Eigenvalues

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 & 1 & 1 \\ 1 & 3-\lambda & -3 & 1 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & -1 & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix}$$

by the rule for the determinant of a block matrix. So

$$|A - \lambda I| = \big((1-\lambda)(3-\lambda)+1\big)(\lambda^2+1) = (\lambda-2)^2(\lambda^2+1).$$

Consider $\lambda = 2$, with multiplicity 2. Eigenvectors?

$$(A-2I)\mathbf{v} = \begin{bmatrix} -1 & -1 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0,$$

gives $c = d = 0$ and we can take $a = 1$, $b = -1$, so $\mathbf{v}_1 = [1, -1, 0, 0]^T$. We need a generalized eigenvector. So we can solve

$$(A-2I)\mathbf{v}_2 = \begin{bmatrix} -1 & -1 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \mathbf{v}_1,$$

which gives $a+b = -1$, and we can let $a = -1$, $b = 0$, and $\mathbf{v}_2 = [-1, 0, 0, 0]^T$. (Alternatively, we could find a non-zero $\mathbf{v}_2$ such that $(A-2I)^2\mathbf{v}_2 = 0$ and then let $\mathbf{v}_1 = (A-2I)\mathbf{v}_2$). The two solutions corresponding to $\lambda = 2$ are

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{2t}, \qquad \mathbf{x}_2(t) = (t\mathbf{v}_1 + \mathbf{v}_2)e^{2t}.$$

With $\lambda = i$, the eigenvector $\mathbf{v}$ satisfies

$$\begin{bmatrix} 1-i & -1 & 1 & 1 \\ 1 & 3-i & -3 & 1 \\ 0 & 0 & -i & 1 \\ 0 & 0 & -1 & -i \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0,$$

which gives $c = 1$, $d = i$, and then $a = (9 - 13i)/25$, $b = (21 + 3i)/25$. Multiplying by 25, we can take

$$\mathbf{v} = \mathbf{v}_3 + i\mathbf{v}_4,$$

where

$$\mathbf{v}_3 = \begin{bmatrix} 9 \\ 21 \\ 25 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix} -13 \\ 3 \\ 0 \\ 25 \end{bmatrix}.$$

The (complex) solution corresponding to $\lambda = i$ is then

$$\mathbf{v}e^{it} = (\mathbf{v}_3 + i\mathbf{v}_4)(\cos t + i \sin t) = \mathbf{x}_3(t) + i\mathbf{x}_4(t),$$

where

$$\mathbf{x}_3(t) = \mathbf{v}_3 \cos t - \mathbf{v}_4 \sin t, \qquad \mathbf{x}_4(t) = \mathbf{v}_3 \sin t + \mathbf{v}_4 \cos t,$$

and the complete, general solution is

$$\mathbf{x}(t) = \sum_{j=1}^{4} c_j \mathbf{x}_j(t).$$

Problem 4

Let

$$F(t, x, \dot{x}) = x^2 + \frac{1}{2} t(t-1)\dot{x}^2,$$

and consider the problem

$$\min \int_2^3 F(t, x(t), \dot{x}(t))\, dt, \quad \text{subj. to} \quad x(2) = 0, \quad x(3) = \log\left(\frac{25}{27}\right). \tag{4}$$

(a) Find the Euler equation (E) for (4).

(b) Show that (E) has a first degree polynomial solution x_1.

(c) Use reduction of order (see EP Exercise 2.2.36) to find another solution x_2 of (E) such that $x_2(t) = v(t)x_1(t)$, $t \in [2, 3]$.

(d) What is the general solution to (E)? Find the unique solution x_* to (E) that satisfies the endpoint conditions in (4).

(e) Decide whether x_* minimizes the integral in (4).

Answer 4(a): The Euler equation (E) is

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0,$$

which in this case is

$$2x - \frac{d}{dt}\left(t(t-1)\dot{x}\right) = 0,$$

or

$$t(t-1)\ddot{x} + (2t-1)\dot{x} - 2x = 0.$$

Answer 4(b): Putting $x_1 = a + bt$ into (E) gives

$$(2t-1)b - 2(a+bt) = 0$$

and so $b + 2a = 0$. We can let $a = -1$, $b = 2$, i.e., $x_1 = 2t - 1$ is one solution to (E).

Answer 4(c): To find a second solution to (E) by reduction of order, we write (E) as

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0,$$

where

$$p(t) = \frac{2t-1}{t(t-1)}, \qquad q(t) = \frac{-2}{t(t-1)}.$$

Then, if $x_2 = vx_1$,

$$x_1\ddot{v} + (2\dot{x}_1 + px_1)\dot{v} = 0,$$

and therefore

$$(2t-1)\ddot{v} + \left(4 + \frac{(2t-1)^2}{t(t-1)}\right)\dot{v} = 0.$$

Letting $u = \dot{v}$,

$$(2t-1)\dot{u} + \left(4 + \frac{(2t-1)^2}{t(t-1)}\right) u = 0.$$

This is a first order equation that is separable:

$$\frac{du}{u} = -\left(\frac{4}{2t-1} + \frac{2t-1}{t(t-1)}\right)\, dt,$$

and the two terms on the right can be integrated individually,

$$\ln(u) = -2\ln(2t-1) - \ln(t(t-1)),$$

and so

$$u = \frac{1}{t(t-1)(2t-1)^2}.$$

We need to integrate this to obtain v. Using partial fractions,

$$u = -\frac{1}{t} + \frac{1}{t-1} - \frac{4}{(2t-1)^2},$$

and so

$$v = -\ln(t) + \ln(t-1) + \frac{2}{2t-1} = \ln\left(1 - \frac{1}{t}\right) + \frac{2}{2t-1}.$$

Thus a second solution is

$$x_2 = vx_1 = (2t-1)\ln\left(1 - \frac{1}{t}\right) + 2.$$

Answer 4(d): The general solution to (E) is

$$x(t) = c_1 x_1(t) + c_2 x_2(t).$$

To find x_* we find c_1 and c_2 from the boundary conitions,

$$\begin{aligned} x_1(2)c_1 + x_2(2)c_2 &= 0, \\ x_1(3)c_1 + x_2(3)c_2 &= \ln(25/27), \end{aligned}$$

and by Cramer's rule,

$$c_1 = -\ln(25/27)x_2(2)/D, \qquad c_2 = \ln(25/27)x_1(2)/D,$$

where

$$D = x_1(2)x_2(3) - x_1(3)x_2(2),$$

and $x_1(2) = 3$, $x_1(3) = 5$,

$$x_2(2) = 3\ln(1/2) + 2, \qquad x_2(3) = 5\ln(2/3) + 2.$$