20$^{\text{th}}$ March, 2017

MAT2440 — Differential equations and control theory

Mandatory assignment

Submission deadline

Thursday 20$^{\text{th}}$ April 2017, 14:30 in the mandatory activity hand-in box, situated on the 7$^{\text{th}}$ floor of Niels Henrik Abels hus.

Instructions

You may write your answers either by hand or on a computer (for instance with LaTeX). All submissions must include the following official front page:

http://www.uio.no/english/studies/admin/compulsory-activities/mn-math-obligforside-eng.pdf

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you understand the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (7$^{\text{th}}$ floor of Niels Henrik Abels hus, e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Problem 1. Let A be the following matrix

$$A = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

where a and b are two real numbers.

a) Determine the exponential $\exp tA$.

b) Find the fundamental solution $\Phi(t; 0)$ of the differential equation

$$\dot{x} = Ax.$$

c) Let $f(t)$ and c be given as

$$f(t) = \begin{pmatrix} e^t \\ 1 \\ 0 \end{pmatrix} \qquad c = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Find the solution of the initial value problem

$$\dot{x} = Ax + f(t) \qquad x(0) = c.$$

Problem 2. Consider the differential equation

$$\dot{x} = Bx$$

where B is the matrix

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Sketch the phase portrait. Which solutions are stable (*i.e.,* bounded when $t \to \infty$) and which are not?

Problem 3. Consider the differential equation

$$\dot{x} = 4/3x^{1/4}. \tag{*}$$

a) For which closed intervals $I = [\alpha, \beta]$ with $0 \leq \alpha < \beta$ is the function $f(x) = x^{1/4}$ Lipschitz? For each I where f is Lipschitz give a Lipschitz constant K for f.

a) For which pairs (t_0, x_0) of real numbers is there a unique solution of (*) with $x(t_0) = x_0$?

b) Let t_0 be a positive real number. Find all solutions of (*) satisfying $x(t_0) = 0$.

Problem 4. a) Let N be a nilpotent $n \times n$-matrix. Show that all eigenvalues of N are zero, and conclude that the characteristic polynomial of N is λ^n. Hint: If v is an eigenvector with eigenvalue λ, then $N^i v = \lambda^i v$.

b) Let A be any $n \times n$-matrix. Show that $\det(\exp A) = \exp(\operatorname{tr} A)$. Hint: Treat the cases when A is semi-simple and nilpotent separately. Then combine.

Problem 5. Given two function $u(t)$ and $v(t)$ both twice continously differentiable in an interval I. Define the *Wronskian determinant* $W(t)$ of the two as the determinant

$$W(t) = \det \begin{pmatrix} u & v \\ u' & v' \end{pmatrix}.$$

a) Show that

$$W(t)' = \det \begin{pmatrix} u & v \\ u'' & v'' \end{pmatrix}.$$

b) Assume that $u(t)$ and $v(t)$ both are solutions of the differential equation

$$y'' + a(t)y' + b(t)y = 0.$$

Show that the Wronskian satisfies the equation

$$W'(t) = -a(t)W(t),$$

and deduce that for a suitable constant C it holds true that

$$W(t) = C \exp(-\int_{t_0}^{t} a(t)dt).$$

c) Conclude that if $(u(t), u'(t))$ and $(v(t), v'(t))$ are linearly independent for one value $t = t_0$, then they are linearly independent for all $t \in I$.