15th May, 2017

MAT2440 — Differential equations and control theory

Solutions to the mandatory assignment

Problem 1. Let A be the following matrix

$$A = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

where a and b are two real numbers.
a) Determine the exponential $\exp tA$.
b) Find the fundamental solution $\Phi(t;0)$ of the differential equation

$$\dot{x} = Ax.$$

c) Let $f(t)$ and c be given as

$$f(t) = \begin{pmatrix} e^t \\ 1 \\ 0 \end{pmatrix} \qquad c = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Find the solution of the initial value problem

$$\dot{x} = Ax + f(t) \qquad x(0) = c.$$

a) The matrix A is given as

$$A = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

where a and b are two real numbers. Clearly the characteristic polynomial equals $(\lambda - 1)^3$, so one is an eigenvalue of multiplicity three (which is the maximal possible multiplicity). The matrix $N = A - I$ is therefore nilpotent; indeed, one has

$$N^2 = \begin{pmatrix} 0 & 0 & ab \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and it is easy to see that $N^3 = 0$. Since I and N commute, this gives $\exp tA = \exp t(I + N) = \exp tI \exp tN = e^t(I + tN + t^2/2 \cdot N^2)$, and thus

$$\exp tA = e^t \begin{pmatrix} 1 & at & abt^2/2 \\ 0 & 1 & bt \\ 0 & 0 & 1 \end{pmatrix}$$

b) The fundamental solution $\Phi(t;0)$ of the differential equation is $\exp tA$ above.

c) Multiplying through by e^{-tA} we bring the equation on the form

$$d/dt(xe^{-At}) = e^{-tA}f(t) = \begin{pmatrix} 1 - e^{-t}at \\ e^{-t} \\ 0 \end{pmatrix}$$

Integrating gives

$$e^{-At}x = \begin{pmatrix} a(te^{-t} + e^{-t} - 1) + t + 1 \\ 1 - e^{-t} \\ 0 \end{pmatrix},$$

and hence

$$x(t) = e^{t} \begin{pmatrix} (a+1)t - a(1 - e^{-t}) + 1 \\ 1 - e^{-t} \\ 0 \end{pmatrix}.$$

Problem 2. Consider the differential equation

$$\dot{x} = Bx$$

where B is the matrix

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Sketch the phase portrait. Which solutions are stable (*i.e.*, bounded when $t \to \infty$) and which are not?

The eigenvalues of B are ± 1 with eigenvectors $(1,1)$ and $(1,-1)$. Hence the solutions are of the form

$$x(t) = c_1 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and $|x(t)|$ tends to infinity with t if $c_1 \neq 0$. Hence the only stable solutions are

$$c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Problem 3. Consider the differential equation

$$\dot{x} = 4/3x^{1/4}. \tag{*}$$

a) For which closed intervals $I = [\alpha, \beta]$ with $0 \leq \alpha < \beta$ is the function $f(x) = x^{1/4}$ Lipschitz? For each I where f is Lipschitz give a Lipschitz constant K for f.
a) For which pairs (t_0, x_0) of real numbers is there a unique solution of (*) with $x(t_0) = x_0$?
b) Let t_0 be a positive real number. Find all solutions of (*) satisfying $x(t_0) = 0$.

a) The function $f(x) = x^{1/4}$ is only defined for $x \geq 0$, and for $x > 0$ its derivative equals $f'(x) = 1/4 \cdot x^{-3/4}$. Assume first that $\alpha > 0$. For any two points x, y from $[\alpha, \beta]$ the Mean Value Theorem gives

$$f(x) - f(y) = 1/4 \cdot c^{-3/4}(x - y),$$

where c lies between x and y and hence is greater than α. The function $t^{-3/4}$ decreases so we obtain

$$|f(x) - f(y)| = 1/4c^{-3/4} |x - y| \leq 1/4 \cdot \alpha^{-3/4} |x - y| .$$

It follows that f is Lipschitz on $[\alpha, \beta]$ with constant $1/4 \cdot \alpha^{-3/4}/4$.

However, f is not Lipschitz on $[0, \beta]$. Again the Mean Value Theorem (this time on the interval $[0, x]$) gives

$$f(x) = 1/4 \cdot c^{-3/4} x$$

where $0 < c < x$, and as $c^{-3/4} \to \infty$ when $x \to 0$, the function f is not Lipschitz on $[0, \beta]$.

b) If $x_0 > 0$ an application of the Picard-Lindlöf theorem, (or if you want, an explicit solution of the equation, see below) shows that there is a unique solution taking the value x_0 at time t_0 for any t_0. However, when $x_0 = 0$ there are two solutions with $x(t_0) = 0$.

Separating the variables, one finds in the region where $x \neq 0$, that

$$3/4 \cdot x^{-1/4} dx = dt,$$

and hence by integration

$$x = (t + c)^{4/3}$$

where c is a constant. One should be careful here; this is a solution only when $t + c \geq 0$, since for $t + c < 0$, it holds true that $(t + c)^{1/3} = -x^{1/4}$.

Additionally, the constant function $x = 0$ is a solution as well.

At the point $t = -c$ one finds that the derivative $4/3 \cdot (t + c)^{1/3}$ of $(t + c)^{4/3}$ vanishes, hence the function $y(t)$ defined by

$$y(t) = \begin{cases} (t + c)^{4/3} & \text{if } t > -c \\ 0 & \text{if } t \leq -c \end{cases} ,$$

is continuously differentiable and solves the differential equation. Hence there are two solutions with $x(-c) = 0$.

Problem 4. a) Let N be a nilpotent $n \times n$-matrix. Show that all eigenvalues of N are zero, and conclude that the characteristic polynomial of N is λ^n. Hint: If v is an eigenvector with eigenvalue λ, then $N^i v = \lambda^i v$.
b) Let A be any $n \times n$-matrix. Show that $\det(\exp A) = \exp(\operatorname{tr} A)$. Hint: Treat the cases when A is semi-simple and nilpotent separately. Then combine.

a) Let v be an eigenvector of N whose corresponding eigenvalue is λ. The $v \neq 0$ by definition of an eigenvector, and $N(v) = \lambda v$. It follows by induction on i that $N^i(v) = \lambda^i v$ for any integer i, but as N is nilpotent, $N^{i_0} = 0$ for some i_0. Hence $\lambda^{i_0} v = 0$ and by consequence $\lambda = 0$ since $v \neq 0$.

b) If A is semi-simple, one may write $A = P\Lambda P^{-1}$ where Λ is diagonal, and since $\operatorname{tr} A = \operatorname{tr} \Lambda$ and

$$\det \exp A = \det P(\exp \Lambda)P^{-1} = \det \Lambda$$

we may assume that A is diagonal. Then

$$\det \exp A = e^{\lambda_1} \dots e^{\lambda_n} = e^{\lambda_1 + \dots + \lambda_n} = \exp \operatorname{tr} A.$$

If N is nilpotent, $\operatorname{tr} N = 0$, and it suffices to see that $\det \exp N = 1$. Now $\exp N = I + P(N)$ where $P(N) = N + 1/2 \cdot N + \dots + 1/n! \cdot N^n$, but $P(N)$ is clearly nilpotent (it can be written as $P(N) = NB$ where N and B commute), and all eigenvalues are zero. It follows that all eigenvalues of $I + PN$ equal one, and hence $\det \exp N = 1$.

In general, write $A = S + N$ where S is semi-simple and N nilpotent and S and N commute. Then

$$\begin{aligned}\det \exp A &= \det \exp(S + N) = \det \exp S \exp N = \det \exp S \det \exp N \\ &= \exp \operatorname{tr} S \exp \operatorname{tr} N = \exp(\operatorname{tr} S + \operatorname{tr} N) = \exp \operatorname{tr}(S + N) = \exp \operatorname{tr} A\end{aligned}$$

Problem 5. Given two function $u(t)$ and $v(t)$ both twice continously differentiable in an interval I. Define the *Wronskian determinant* $W(t)$ of the two as the determinant

$$W(t) = \det \begin{pmatrix} u & v \\ u' & v' \end{pmatrix}.$$

a) Show that

$$W(t)' = \det \begin{pmatrix} u & v \\ u'' & v'' \end{pmatrix}.$$

b) Assume that $u(t)$ and $v(t)$ both are solutions of the differential equation

$$y'' + a(t)y' + b(t)y = 0.$$

Show that the Wronskian satisfies the equation

$$W'(t) = -a(t)W(t),$$

and deduce that for a suitable constant C it holds true that

$$W(t) = C \exp(-\int_{t_0}^{t} a(t)dt).$$

c) Conclude that if $(u(t), u'(t))$ and $(v(t), v'(t))$ are linearly independent for one value $t = t_0$, then they are linearly independent for all $t \in I$.

a) One finds

$$W' = (uv' - vu')' = u'v' + uv'' - v'u' - vu'' = uv'' - u''v = \det \begin{pmatrix} u & v \\ u'' & v'' \end{pmatrix}.$$

b) One finds if $u'' = -au' - bu$ and $v'' = -av' - bv$ that

$$W = \det \begin{pmatrix} u & v \\ -au' - u & -av' - bv \end{pmatrix} = \det \begin{pmatrix} u & v \\ -au' & -av' \end{pmatrix} = -aW.$$

By the classical formula for the solutions of a first order equation, it follows that

$$W(t) = C \exp(-\int_{t_0}^{t} a(t)dt).$$

c) If $(u(t_0), u'(t_0))$ and $(v(t_0), v'(t_0))$ are independent, one sees that for the constant C above one has

$$W(t_0) = C \neq 0,$$

hence since the exponential function never vanishes, it follows that $W(t) \neq 0$ for any t and hence $(u(t), u'(t))$ and $(v(t), v'(t))$ are linearly independent.