UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT2700 — Introduction to mathematical finance and investment theory.

Day of examination: Monday, December 13, 2010.

Examination hours: 14.30 – 18.30.

This problem set consists of 2 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Consider the following 1–period market, with $r = \frac{1}{9}$, $K = 3$, $N = 1$,

ω	$S_1(0)$	$S_1(1,\omega)$	$P(\omega)$
$\omega = \omega_1$	2	$\frac{10}{3}$	$\frac{1}{3}$
$\omega = \omega_2$	2	$\frac{10}{9}$	$\frac{1}{3}$
$\omega = \omega_3$	2	$\frac{50}{9}$	$\frac{1}{3}$

(a) [10 units] Find all risk neutral probability measures Q for this market.

(b) [10 units] Explain why the market has no arbitrage and why it is not complete.

(c) [10 units] We want to add a fictitious security with prices $S_2(0)$ and $S_2(1,\omega)$ to the market, in order to make it complete and still remain free from arbitrage. Explain why $S_2(1) = (5, 1, 1)$ is a possible choice.

Problem 2

Consider the 2–period market with the following prices:

ω	$S_1(0)$	$S(1,\omega)$	$S(2,\omega)$	$P(\omega)$
$\omega = \omega_1$	3	1	1	$\frac{1}{3}$
$\omega = \omega_2$	3	5	4	$\frac{1}{3}$
$\omega = \omega_3$	3	5	7	$\frac{1}{3}$

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We have $r = 0$, $K = 3$ and $N = T = 2$.

(a) [10 units] For $t = 1, 2$, we let $\mathcal{F}_t$ be the algebra of subjects of Ω generated by the random variables $S(i, \cdot)$, $i \leq t$. Find $\mathcal{F}_1$ and $\mathcal{F}_2$.

(b) [10 units] Let $X = (X(\omega_1), X(\omega_2), X(\omega_3)) = (3, 1, 0)$. Is X measurable with respect to $\mathcal{F}_1$?

(c) [10 units] Find $\mathbb{E}[X|\mathcal{F}_1]$.

(d) [10 units] Show that $Q = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ is a martingale measure for this market.

(e) [10 units] Find a predictable, self–financing portfolio H which replicates the random variable X at time $t = 2$, where X is as in (b).

(f) [20 units] Let $\nu \in \mathbb{R}$ be a given initial wealth. We want to use dynamic programming to study the problem:

$$\begin{array}{ll} \max_{H \in \mathbb{H}} & \mathbb{E}[u(V_2^{(H)})] \\ \text{subject to} & V_0 = \nu. \end{array}$$

Here $u(x) = -\exp(-x)$, $x \in \mathbb{R}$, is the exponential utility function, $\mathbb{H}$ is the space of self–financing, predictable portfolios H and $V_t^{(H)}$ is the wealth at time t generated by the portfolio H; $t = 0, 1, 2$.

The dynamic programming principle states that

$$U_t(W) = \max_H \mathbb{E}[U_{t+1}(B_{t+1}\{\frac{W}{B_t} + H \cdot \Delta S^*(t+1)\})|\mathcal{F}_t],$$

where $U_t(W)$ is the maximum expected utility of the wealth at time T, given that the wealth at time t is W and given $\mathcal{F}_t$; $t = 0, 1$.

Find $U_1(W)$, $U_0(W)$ and the corresponding optimal portfolio $H_1(2, \omega)$, $H_1(1)$ and $H_0(2, \omega)$, $H_0(1)$; $\omega \in \Omega = \{\omega_1, \omega_2, \omega_3\}$.

END OF EXAM.