UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in:	MAT2700 —	Introduction to mathematical finance and investment theory.
Day of examination:	Monday, December 12, 2011.	
Examination hours:	14.30 – 18.30.	

This problem set consists of 3 pages.

Appendices:	None.
Permitted aids:	None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Remember to give reasons for your answers.

Problem 1.
Consider a one period financial market with $N = 2$ securities, $K = 3$ scenarios, interest rate $r = 0$ and return processes R_n and probability measure $\mathbb{P}$ as given in the table below:

ω	$R_1(\omega)$	$R_2(\omega)$	$P(\omega)$
ω_1	0.2	0.15	1/3
ω_2	-0.2	0	1/3
ω_3	0.05	-0.1	1/3

Let $F_n = \frac{H_n S_n(0)}{V_0}$ be the fraction of the total wealth invested in security n at time 0. Then

$$V_1 = \nu\Big(1 + \sum_{n=1}^{2} F_n R_n\Big)$$

is the value at time $T = 1$ corresponding to F_1, F_2 when the initial value is $\nu > 0$.
Suppose shortselling is not allowed. This means that we are required to have

$$F = (F_1, F_2) \in \mathbb{K},$$

where $\mathbb{K} = \{(x_1, x_2) \in \mathbb{R}^2, x_1 \geq 0, x_2 \geq 0\}$.
We want to maximize the expected utility of the terminal wealth under this constraint, i.e., solve the problem

$$J_0(F) = \max_{F \in \mathbb{K}} \mathbb{E}[-\exp(-V_1)], \tag{1}$$

where $V_1 = \nu\Big(1 + \sum_{n=1}^{2} F_n R_n\Big)$ is wealth at time $T = 1$ and

$$u(x) = -\exp(-x), \qquad x \in \mathbb{R},$$

is the exponential utility. To this end, we proceed as follows:

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1a) (Step 1) Find the support function δ of $\mathbb{K}$, defined by

$$\delta(\kappa) = \sup_{F \in \mathbb{K}} \{-F \cdot \kappa\}, \qquad \kappa \in \mathbb{R}^2,$$

where $F \cdot \kappa = F_1\kappa_1 + F_2\kappa_2$ is the scalar product of F and κ.

1b) (Step 2) For $\kappa \in \tilde{\mathbb{K}} = \{\kappa;\ \delta(\kappa) < \infty\}$ define the auxiliary market $\mathcal{M}_\kappa$ by specifying the new interest rate to be

$$r_\kappa := r + \delta(\kappa) = \delta(\kappa)$$

and the new returns to be

$$R_n^\kappa := R_n + \delta(\kappa) + \kappa_n, \ \ n = 1, 2.$$

Verify that

$$\begin{aligned} Q_\kappa &= (Q_\kappa(\omega_1), Q_\kappa(\omega_2), Q_\kappa(\omega_3)) \\ &:= \frac{1}{31}(8 - 40\kappa_1 - 100\kappa_2, 11 + 100\kappa_1 - 60\kappa_2, 12 - 60\kappa_1 + 160\kappa_2) \end{aligned}$$

is a risk neutral probability measure for $\mathcal{M}_\kappa$, provided that $\kappa \in \tilde{\mathbb{K}}$ and $40\kappa_1 + 100\kappa_2 < 8$.

1c) (Step 3) In the following we assume that the markets $\mathcal{M}_\kappa$, with $\kappa \in \tilde{\mathbb{K}}$ and $40\kappa_1 + 100\kappa_2 < 8$, are complete.

For each $\kappa \in \tilde{\mathbb{K}}$ we proceed to solve the *unconstrained* problem

$$J_\kappa(\nu) = \max_{F \in \mathbb{R}^2} \mathbb{E}[-\exp(-V_1^{(\kappa)})],$$

where

$$V_1^{(\kappa)} = \nu\Big(1 + r_\kappa + \sum_{n=1}^{2} F_n(R_n^{(\kappa)} - r_\kappa)\Big)$$

is the value at $T = 1$ in the market $\mathcal{M}_\kappa$ corresponding to the portfolio $F = (F_1, F_2) \in \mathbb{R}^2$.

Use the risk neutral probability approach to find the optimal terminal wealth

$$\hat{W}_\kappa = \hat{V}_1^{(\kappa)}$$

for this problem, in terms of $L_\kappa(\omega) = \frac{Q_\kappa(\omega)}{\mathbb{P}(\omega)}$, $\omega = \omega_1, \omega_2, \omega_3$.

1d) (Step 4) Finally, we proceed to find the optimal terminal wealth $\hat{W}_\kappa = \hat{V}_1^{(\kappa)}$ of the original unconstrained problem (1) by minimizing

$$J_\kappa(\nu) = \mathbb{E}[-\exp(-\hat{W}_\kappa)]$$

over all $\kappa = (\kappa_1, \kappa_2) \in \tilde{\mathbb{K}}$. Write down the first order equations for the minimizing $\kappa_1 = \hat{\kappa}_1$, $\kappa_2 = \hat{\kappa}_2$.

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Problem 2.

Consider the following 2–period market, with $N = 1$, $K = 4$, interest rate $r = 0$, probability measure $\mathbb{P} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and prices $S(t, \omega_i)$ given by the table below:

ω	$S(0)$	$S(1,\omega)$	$S(2,\omega)$
ω_1	3	4	7
ω_2	3	2	3
ω_3	3	4	3
ω_4	3	2	1

Let $\mathcal{F}_t$ be the σ–algebra generated by $S(u, \cdot)$, $u \le t$.

2a) Find $\mathcal{F}_1$ and $\mathcal{F}_2$.

2b) Let $Q = (Q(\omega_1), Q(\omega_2), Q(\omega_3), Q(\omega_4)) = (\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4})$. Show that Q is a martingale measure for this market.

2c) Find $\mathbb{E}_Q[Y|\mathcal{F}_1]$ when $Y(\omega_1) = 36$, $Y(\omega_2) = 9$, $Y(\omega_3) = 4$, $Y(\omega_4) = 9$.

2d) Find all martingale measures for this market.

2e) Find a (self–financing, predictable) replicating portfolio $H = (H_0(t), H_1(t))$, $t = 1, 2$, for the contingent claim Y given in 2c) above.

We want to use the martingale method to solve the following optimal portfolio problem:

$$\begin{cases} \text{maximize} & \mathbb{E}[u(V_2^{(H)})] \qquad \text{over all } H \in \mathbb{H}, \\ \text{subject to} & V_0^{(H)} = \nu, \text{ a given real number.} \end{cases} \tag{2}$$

Here $u(x) = 2x^{\frac{1}{2}}$ and $V_2^{(H)}$ is the value process at time $T = 2$ obtained by using the portfolio $H \in \mathbb{H}$, where $\mathbb{H}$ is the set of all self-financing predictable portfolios. Let $\hat{H}$ denote an optimal portfolio for this problem.

2f) Explain why the optimal value $V_2^{(\hat{H})}$ at time 2 is given as the solution $\hat{W} = (\hat{W}(\omega_1), \hat{W}(\omega_2), \hat{W}(\omega_3), \hat{W}(\omega_4))$ of the following problem

$$\begin{cases} \text{maximize} & \mathbb{E}[u(W)] \\ \text{over all} & W = (W(\omega_1), W(\omega_2), W(\omega_3), W(\omega_4)) \\ \text{subject to} & \mathbb{E}_Q[W] = \nu, \end{cases} \tag{3}$$

where Q is the measure in 2b).

2g) Solve the problem (3) by using Lagrange multipliers.

2h) Find the optimal portfolio $\hat{H}$ for the problem (2). (Hint: You may use the result of 2e) above.)

END